

FINITE ADAPTABILITY OF COUNTEREXAMPLE LIBRARIES

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ABSTRACT

How many fixed noise witnesses cover every measurement batch compatible with a pair of model predictions? Under diagonal error, requests and witnesses share a knapsack constraint. Identical best coverage and demand sizes within a factor of two can still yield exponentially different library sizes. We separate information lost by grouping requests from information lost by grouping completions. An input-level test computes the unique coarsest partition preserving compatible demand, constructing groups within a cap or certifying its insufficiency. Optimistic and robust type programs then bracket the full fractional cover with a computable loss guarantee; partial-demand counts supply lower certificates when stability fails. Distinct-weight families isolate joint information beyond optimal packing and strengthened counting, and quantify unavoidable aggregation loss. On all 164 small instances, canonical grouping expands eight-group bracket applicability from 39 to 157; 107 of these 157 brackets have ratios at most 1.25. All 48 optimized witnesses require more than eight stable groups, exposing a substantial transfer boundary. Checked greedy libraries and matched-time counting/packing controls distinguish stronger certificates from changed integer decisions and from runtime gains.

1 INTRODUCTION

For a fixed pair of model predictions, how many noise witnesses explain every compatible measurement batch? This is a finite-adaptability question: recourse is restricted to preselected actions (Hanasusanto et al., 2015). With finite uncertainty, minimum policy count is already set cover, and demand divided by best single-policy coverage is a standard lower bound (Kurtz, 2026). We ask what demand composition reveals beyond this count and how much of that information grouping preserves.

Requests and witnesses share the same feasibility constraint. Under diagonal ellipsoidal error, a measurement batch is compatible with two fixed model predictions when its coordinate weights sum to at most a threshold. A concrete noise witness answers the batches inside a feasible coordinate set. The same incidence describes reuse of satisfying assignments for Boolean knapsack queries, an operation modeled on concrete-assignment caches (Cadaru et al., 2008; Palikareva & Cadaru, 2013). A fresh witness costs only $O(k)$ here: the question is retained coverage under a capacity constraint, independently of the cost of answering a request another way.

Grouping can lose information in two ways: it can mix compatible and incompatible requests, or feasible and infeasible completions. Preserving demand alone does not prevent the second loss. We compute the unique coarsest demand-preserving partition from threshold-pivotal subsets, using classical hypergraph twins and threshold-family preorders. This test either constructs a partition within the group cap or certifies its insufficiency. Once demand is preserved, optimistic and robust type LPs bracket the remaining loss through a computable contraction factor. Conservative demand counts instead supply lower certificates when the cap is insufficient.

Structural separations explain why joint information matters. With identical best coverage and nearly matched demand, library sizes can differ exponentially. Seven joint constraints on distinct-weight inputs beat optimal pairwise packing by a growing factor and every uniform-price union of those demand types by a constant factor. A separate three-row construction makes aggregation loss explicit. Experiments then distinguish stronger certificates from decisions: canonical grouping

expands eight-group applicability on the small-instance set from 39/164 to 157/164, but all 48 optimized witnesses require more groups. Matched-time controls and checked greedy libraries delimit the resulting lower certificates’ additional value.

2 RELATED WORK

Finite adaptability and policy count. K -adaptability restricts recourse to preselected actions (Hanasusanto et al., 2015). For finite uncertainty, Kurtz (2026, Section 5) gives the set-cover formulation, residual-coverage greedy approximation and counting lower bound. We implement this greedy integer upper; our additional results are matched-statistics separations and certified aggregation under shared knapsack geometry.

Covering and aggregation. Fractional covers, covering designs and type counting are established (Lovász, 1975; Gordon et al., 1996; Csiszár, 1998). Formula-size versus incompatible-point gaps (Hellerstein & Kletenik, 2013) and lifted LPs (Mladenov et al., 2012) provide related abstractions. Transposition twins (Babai, 2017, Definition 2.5.6), threshold-family preorders (Klivans & Reiner, 2008), and knapsack counting (Halman, 2016) are inherited tools. We distinguish demand symmetry from completion ambiguity, compute stable groups from weights, and certify cover loss. Our rounding is not a new counting scheme.

Counterexamples and robust design. KLEE reuses concrete assignments (Cadaru et al., 2008; Palikareva & Cadaru, 2013); our cache consequence concerns that specialized operation. Bounded-error recovery motivates the geometric model (Donoho, 1994; Foucart & Liao, 2024). Hitting sets, CEGIS, Benders and symbolic compilation retain richer objects than concrete libraries. Appendix R compares their contracts and robust-design connections; our policy counts do not bound those representations.

3 METHODOLOGY

3.1 COUNTEREXAMPLE LIBRARIES AS FINITE ADAPTABILITY

Consider two models with difference d , whose full prediction difference is $z \in \mathbb{R}^E$ and whose target difference is $\Delta \geq 0$. Under ellipsoidal observation error $\Xi = \{\xi : \xi^\top \Sigma^{-1} \xi \leq c^2\}$, a measurement batch B cannot distinguish them exactly when $Q_z(B) = z_B^\top \Sigma_{BB}^{-1} z_B \leq T = 4c^2$. A legal noise difference $h \in \Xi - \Xi$ certifies ambiguity only on batches inside its cancellation set $C(d, h) = \{e : h_e = z_e\}$. Projection instead permits a different legal h for each batch (Figure 1). Appendix A derives this fixed-pair evidence from worst-case posterior risk.

For diagonal Σ , write $w_e = z_e^2 / \Sigma_{ee} \geq 0$. Both compatible requests and legal cancellation sets have the same capacity constraint:

$$\mathcal{U}_k = \{\xi \in \{0, 1\}^E : \mathbf{1}^\top \xi = k, w^\top \xi \leq T\}, \quad \mathcal{V} = \{y \in \{0, 1\}^E : w^\top y \leq T\}. \quad (1)$$

A library of M completions is sufficient precisely when

$$\exists y^1, \dots, y^M \in \mathcal{V} \quad \forall \xi \in \mathcal{U}_k \quad \exists j \in [M] : y^j \geq \xi. \quad (2)$$

This is the feasibility form of M -adaptability: choose finitely many recourse actions before the request, then select one after it (Hanasusanto et al., 2015). Its minimum M is our *completion complexity* $\chi_k(z)$. A feasible y gives a legal completion $h = \text{diag}(z)y$; every actual completion supplies a feasible cancellation set. Zero coordinates may enlarge cancellation but do not alter the optimum. Set $\chi_k(z) = 0$ for empty demand.

Capacity and computation. The identity policy $y(\xi) = \xi$ costs $O(k)$ arithmetic ($O(E)$ for dense output). A cached Boolean model with support C answers the query requiring $x_e = 1$ on B exactly when $B \subseteq C$, giving the same incidence. This specialization does not make caching necessary or a miss expensive. Both families in (1) are discrete; convexifying them changes the problem.

Figure 1: **From a counterexample to its completion complexity.** (a) A fixed legal noise difference h retains only batches contained in its cancellation set; the projected certificate retains (z, Δ) and allows a different legal completion for each batch. The illustrated completion covers three of the twelve compatible pairs in (b); projection covers all twelve. (b) With three light coordinates of weight 1 (green), three heavy coordinates of weight 3 (orange), $k = 2$ and $T = 4$, one completion covers all light pairs, while each of the nine mixed pairs exhausts the budget and needs its own. Thus $\chi_2(z) = 10$; heavy pairs are incompatible. (c) The union test characterizes one-completion export, and the grouped profile bounds larger libraries. These are representation guarantees, not search-time comparisons.

3.2 WHY MAXIMUM COVERAGE CANNOT DECIDE LIBRARY SIZE

Let $\mathcal{F}_{z,k} = \{B : |B| = k, \sum_{e \in B} w_e \leq T\}$, $N = |\mathcal{F}_{z,k}|$, and m be the largest feasible cancellation cardinality. The standard certificate is $\chi_k(z) \geq N / \binom{m}{k}$ (Kurtz, 2026, Remark 31). For nonempty demand, one completion suffices exactly when its union $J_{z,k}$ is feasible. Testing whether each coordinate joins its $k - 1$ cheapest others computes this union in $O(E \log E)$ (Proposition 1).

Matched statistics, exponentially different capacities. For even n , take $E = 4n$, $k = n$, $T = 2n$. Compare homogeneous weights one with $2n$ weights one and $2n$ weights three. Both have best coverage $\binom{2n}{n}$, and demand sizes differ by at most a factor of two, but

$$\lim_{n \rightarrow \infty, 2|n} \frac{1}{4n} \log \frac{\chi_{\text{het}}}{\chi_{\text{hom}}} = \frac{1}{2} \log 2. \quad (3)$$

Each of the $\binom{2n}{n/2}^2$ heterogeneous batches with equal numbers of light and heavy coordinates exhausts the budget and needs its own completion. Homogeneous batches share larger completions; classical covering bounds give the matching rate (Theorem 7). Both families are rank-one realizable with bounded normalized geometry. At $E = 64$, their certified bounds are respectively above 1.10×10^{14} and below 5.29×10^7 . The gap survives distinct weights in fixed-width intervals, with liminf exponent above .3150, though total demand then loses the factor-two match (Theorem 8).

3.3 COMPOSITION CERTIFICATES WITHOUT EXACT WEIGHT SYMMETRY

A small partition can expose composition information without enumerating the full cover. Write τ^* for the full fractional set-cover optimum. Lifted LPs aggregate structured rows and columns (Mladenov et al., 2012); here we use a lower relaxation and bound the information it loses.

Counting demand and feasible completion types. Partition the coordinates into G groups I_g of sizes E_g . For a set S , its type is the vector $(|S \cap I_g|)_{g=1}^G$. Let n_b count the members of $\mathcal{F}_{z,k}$ of type b ; retain only rows with $n_b > 0$. Sort the weights within group g , and let $s_g(j)$ be the sum of its j smallest weights, with $s_g(0) = 0$. The types of coordinate sets that can be canceled are exactly

$$\mathcal{C}_{\mathcal{P}} = \{c : 0 \leq c_g \leq E_g, \sum_g s_g(c_g) \leq T\}. \quad (4)$$

Incidental cancellation of zero-prediction coordinates can enlarge a set; the realizable superset only strengthens coverage. Every feasible coordinate set of type c contains exactly $K_{bc} = \prod_g \binom{c_g}{b_g}$ compatible batches of type b . With $a_{bc} = K_{bc}/n_b$, define the lower relaxation

$$D_{\mathcal{P}} = \min_{x \geq 0} \left\{ \sum_c x_c : \sum_c a_{bc} x_c \geq 1 \text{ for every } b \right\}. \quad (5)$$

These constraints preserve counts within a type, not coverage of each individual batch. Thus fractional solutions need not give realizable libraries under unequal weights.

Theorem 1 (Partition hierarchy). *For nonnegative diagonal weights and a nonempty compatibility family, $D_{\mathcal{P}} \leq \chi_k(z)$. If $\mathcal{P}'$ refines $\mathcal{P}$, then $D_{\mathcal{P}} \leq D_{\mathcal{P}'}$. With one group, $D_{\mathcal{P}} = N/\binom{m}{k}$, where $N = |\mathcal{F}_{z,k}|$ and m is the maximum feasible cancellation cardinality. With singleton groups, $D_{\mathcal{P}}$ is the full fractional set-cover optimum. With equal weights inside every group, it equals the exact orbit LP.*

Summing batch constraints proves the lower bound; Vandermonde’s identity proves refinement (Appendix C).

Certification without a batch list. Exact counts n_b need not be available. Let $s_g^+(j)$ be the sum of the j largest weights in group g . A type b with $\sum_g s_g^+(b_g) \leq T$ certifies all $N_b = \prod_g \binom{E_g}{b_g}$ of its batches as compatible. Retain these fully compatible types, replace n_b by N_b in the dual, and keep the same minimum-prefix completion types (4). Any feasible dual of value D^- still proves $\chi_k(z) \geq D^-$ and the omission guarantee (8). Its distribution can be sampled by choosing a type and then uniform coordinate subsets within groups. No actual compatible batches are enumerated or counted.

Verification is polynomial for fixed G without a batch list (Proposition 4); this does not bound dense solver time. Before solving, test the one-row dual $q^- = \max_b N_b / \max_c K_{bc}$; for counted types use $q_{\mathcal{P}} = \max_b n_b / \max_c K_{bc}$. Combining types matters only when this simpler bound leaves a decision open. Unlike the counted LP, these one-row bounds need not increase under refinement.

Retaining partially compatible demand types. When no affordable stable partition exists, discarding every ambiguous demand type can leave a weak lower certificate. We instead count a certified subfamily, while leaving completion feasibility unchanged. For $T > 0$ and an integer resolution Q , let $a_e^+ = \lceil Qw_e/T \rceil$ and $a_e^- = \lfloor Qw_e/T \rfloor$. Cardinality-aware knapsack dynamic programming counts each type under $\sum_{e \in B} a_e^+ \leq Q$ and $\sum_{e \in B} a_e^- \leq Q$, giving $n_b^- \leq n_b \leq n_b^+$. Fully compatible types retain their exact count N_b . In the equivalent unnormalized dual,

$$D(r) = \max_{\lambda \geq 0} \left\{ \sum_b r_b \lambda_b : \sum_b K_{bc} \lambda_b \leq 1 \quad (c \in \mathcal{C}_{\mathcal{P}}) \right\}, \quad D(n^-) \leq \tau^*. \quad (6)$$

The columns still use the original weights in (4). Thus even a partial demand count gives a valid lower certificate. Doubling Q nests the counted subfamilies; it does not ensure convergence at exact threshold equality. This uses classical knapsack counting machinery (Halman, 2016), not a new counting approximation scheme. Before the joint LP, test the same-count single-row bound $\max_b n_b^- / \max_c K_{bc}$ and aggregate counting. Appendix F gives the implementation and failure boundary.

When do a few groups retain cover information? The lower hierarchy alone gives no approximation guarantee. Define a type as *robust* when $\sum_g s_g^+(c_g) \leq T$: all its realizations are feasible.

Suppose every size- k type that passes the minimum-prefix test also passes the maximum-prefix test. Call this *demand stability*. It is testable from weights, implies $n_b = N_b$, and still permits completion types with both feasible and infeasible realizations.

For each maximal feasible type c , form the multiset of the c_g largest weights in each group. Let $m_c = \sum_g c_g$ and let r_c be the largest number of its largest entries whose sum is at most T . Keep only $m_c \geq k$, and define $\rho = \min_c \binom{r_c}{k} / \binom{m_c}{k}$. Let $U_{\mathcal{P}}$ be the type LP restricted to robust columns, with coefficients K_{bc}/N_b .

Theorem 2 (A computable grouping-loss bound). *For nonempty demand under demand stability, $r_c \geq k$, $\rho > 0$, and*

$$D_{\mathcal{P}} \leq \tau^* \leq U_{\mathcal{P}} \leq D_{\mathcal{P}}/\rho, \quad \chi_k(z) \leq \min\{N, \lceil (1 + \log N) D_{\mathcal{P}}/\rho \rceil\}. \quad (7)$$

If $k(m_c - r_c)/m_c \leq \varepsilon < 1$ for every such c , then $\rho \geq 1 - \varepsilon$. The demand test and retention factor require only type enumeration and prefix order statistics, with polynomial bit cost for fixed G .

Uniformly retaining r_c of m_c abstract labels gives robust types and preserves each size- k subset with probability $\rho_c = \binom{r_c}{k} / \binom{m_c}{k}$. Replacing column mass x_c by this mixture with mass x_c/ρ_c preserves coverage. Symmetrization gives a fractional cover and standard rounding the integer bound (Appendix C.3). A deterministic contracted type can instead lose a whole demand row.

The direct interval $[D_{\mathcal{P}}, U_{\mathcal{P}}]$ can be much tighter than the worst-column conversion (Appendix K.1). If demand stability fails, the lower certificate remains valid, but the full-family upper guarantee is withheld; Appendix C.3 accounts for uncertified demands with an explicit residual term.

The dependence on ρ is order-tight: already at $k = 2$, one group with stable demand and distinct weights can have $\tau^*/D_{\mathcal{P}} \geq 1/(5\rho)$ (Appendix C.3). Demand stability alone therefore cannot ensure a constant approximation.

What a lower certificate guarantees. A feasible dual $y \geq 0$ of (5), of value $D = \sum_b y_b > 0$, defines a distribution: choose type b with probability y_b/D , then a uniform compatible batch of that type. Every completion covers mass at most $1/D$. Thus any library of at most M completions omits

$$\Pr\{B \text{ compatible but uncovered}\} \geq \max(0, 1 - M/D). \quad (8)$$

The same guarantee holds conditionally for a cache that changes after every query: if each next query is an independent draw from this dual distribution, any history-dependent cache with at most M pre-query models misses with conditional probability at least $\max(0, 1 - M/D)$. Conditioning on the cache and applying a union bound proves this consequence (Appendix J). The guarantee concerns concrete-model reuse under that distribution, rather than total solver work or overall robust risk.

3.4 CHOOSING GROUPS THAT PRESERVE COMPATIBLE DEMAND

To distinguish partition error from an insufficient group cap, call two coordinates *demand twins* when exchanging them preserves the size- k compatibility family. This is the classical transposition notion of hypergraph twins (Babai, 2017, Definition 2.5.6). For sorted threshold families, these are also equivalence classes of the classical vicinal preorder (Klivans & Reiner, 2008, Section 2.3). Adjacent threshold pivots compute the classes directly from weights.

Theorem 3 (Minimum stable grouping). *Let $1 \leq k \leq E$ and sort $w_1 \leq \dots \leq w_E$. The demand-twin classes are contiguous and form the unique coarsest demand-stable partition. An adjacent cut $i, i + 1$ is required exactly when some $R \in \binom{[E] \setminus \{i, i+1\}}{k-1}$ satisfies*

$$T - w_{i+1} < \sum_{j \in R} w_j \leq T - w_i. \quad (9)$$

All required cuts, one witness per cut, and the minimum group count G_{DS} can be computed in $O(E \log E + \binom{E}{k-1}(k + \log E))$ arithmetic operations.

The algorithm enumerates each $(k-1)$ -set R once, binary-searches the last weight at most $T - w(R)$, and marks that adjacent cut if neither endpoint belongs to R . Splitting at the marked cuts gives the

minimum partition. Appendix D proves completeness, including repeated and zero weights. The cost is polynomial for fixed k , and can be prohibitive when k varies.

This yields a deterministic grouping rule. Given a group cap $G_{\max}$, compute the required cuts. Finding $G_{\max}$ distinct required cuts already proves $G_{\text{DS}} > G_{\max}$, so a refusal can stop before computing the complete partition. In this case, a full-family certificate using only robust column orbits is unavailable at that cap; use a lower certificate or explicit boundary repair. Otherwise start from the canonical partition and, if desired, split its largest block at the median until the cap, breaking ties from left to right. Every such refinement remains demand-stable, increases $D_{\mathcal{P}}$ and decreases $U_{\mathcal{P}}$ weakly. One may stop when $U_{\mathcal{P}}/D_{\mathcal{P}} \leq 1 + \varepsilon$ certifies the requested fractional precision. A small G_{DS} guarantees applicability, not a tight bracket. Conversely, even weights in an arbitrarily short interval can require $E - 1$ stable groups.

3.5 JOINT INFORMATION BEYOND COUNTING AND PACKING

For an explicit family, a maximal completion is mandatory if some batch has no other covering column. Charging the F mandatory columns and removing their covered demands decomposes integer cover, fractional cover and packing optima as F plus the residual optimum (Lemma 1). We include this standard preprocessing and recompute counting on actual residual incidences; removing demands can destroy type symmetry.

Joint prices add information even with exact type invariance. For $k = 6$, take $n \geq 10$, $4n$ light and n heavy distinct weights near 1 and $3/2$, with $T = 10.51$. Seven demand rows and eight maximal completion types give an exact fractional cover $D_n = \Theta(n^6)$, while optimal pairwise packing is at most $\binom{5n}{5}/6 = O(n^5)$: incompatible batches cannot share a five-subset because every seven-set is feasible. No completion is mandatory. Beyond this growing packing gap, D_n exceeds the best uniform-price bound over *every union of the seven demand types* by a factor tending to approximately 1.558; that control includes aggregate and single-row counting. Every refined single-row bound is also weaker (Proposition 6). The control does not include arbitrary subsets of individual demands. This exact-symmetry construction isolates joint pricing; the next construction isolates aggregation loss.

Joint information without forced completions. For every even $n \geq 6$, a second distinct-weight two-group family has stable demand and ambiguous completions, but every batch belongs to at least n maximal completions. No column is forced. Its exact type values are

$$D_{\mathcal{P}} = \frac{7n^2 - 2n}{12}, \quad U_{\mathcal{P}} = \frac{2n^2 - n}{3}, \quad \omega_{\text{pair}} = \frac{n^2}{2} - 1, \quad q_{\mathcal{P}} = \frac{n^2}{2}, \quad (10)$$

where ω_{pair} counts a largest collection of batches with pairwise infeasible unions. Aggregate counting and every single-type bound under a refinement of the two groups are no larger than $q_{\mathcal{P}}$. Three rows explain the joint bound. With n light and n heavy coordinates and $H = \binom{n}{2}$, the demand counts are (H, n^2, H) . The optimistic columns $(4, 0)$, $(2, 1)$, $(1, 2)$ have coverage matrix

$$K^- = \begin{pmatrix} 6 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda^- = (1/6, 5/12, 1/6). \quad (11)$$

Every column has dual load one. The mixed row alone gives $n^2/2$, whereas the three rows jointly give $(7n^2 - 2n)/12$. Robustness replaces the ambiguous four-light column by three lights, changing only the upper-left entry from six to three; the uniform per-demand dual $(1/3, 1/3, 1/3)$ then gives $U_{\mathcal{P}}$. Thus $D_{\mathcal{P}}/q_{\mathcal{P}} \rightarrow 7/6$ even after mandatory-column preprocessing. Moreover, $U_{\mathcal{P}} - (2n - 3)/3 \leq \tau^* \leq U_{\mathcal{P}}$, so $\tau^*/D_{\mathcal{P}} \rightarrow 8/7$. The weighted column-contraction bound attains $U_{\mathcal{P}}$ here. Appendix G.2 gives matching primal-dual certificates and the packing proof.

Checked certificates and compiler decisions. Numerical LP output supplies feasible certificates, which need not attain the theoretical optima. A checked dual value L rejects capacity M when $L > M$; dividing L by ρ does not certify an upper. Under demand stability, an optimistic primal of cost V gives $\tau^* \leq V/\rho$, and a robust primal of cost W gives $\tau^* \leq W$. Only a checked integer library of size at most M permits export; otherwise the projected predicate remains (Table 3).

Grouping	Valid / 164	$D = U$	$U/D \leq 1.25$	$U/D \leq 2$	Median U/D	Valid ms
Fixed 4	19	12	12	14	1.000	2.55
Canonical 4	78	37	40	53	1.231	2.82
Fixed 8	39	24	30	38	1.000	3.83
Canonical 8	157	79	107	150	1.000	5.16

Table 1: **Constructive grouping and its refinement ablation.** Blue marks increased applicability at the same cap. Counts retain all 164 conditions. Ratio and time medians use each method’s valid subset; “Valid ms” charges construction, both LPs and exact checks, and is not a cross-method speed ratio. The unfilled canonical partition has 157 valid brackets, 60 exact and 73 within 1.25, at median 2.94 ms.

4 EXPERIMENTAL RESULTS

Reported D and U are exactly checked feasible dual and primal values; matching bounds establish optimality. Small-instance analyses are exploratory. Input designs for stress, optimized-witness and size-sweep evaluations are specified before generation. Failed and capped attempts are included.

4.1 DOES CHOOSING THE PARTITION REPAIR DEMAND INSTABILITY?

The small-instance set has 140 continuous witnesses from 48 diagonal tasks and 12 seed streams, plus 24 Wine conditions from four streams ($E = 8, 12, 13, k = 2, 3$). We compare fixed equal-sized weight partitions with the canonical partition followed by largest-block median splits up to the same cap. Three runs rotate method order; every clock includes group construction, both LPs and rational checks.

At eight groups, fixed partitioning gives 39 full-family brackets; canonical grouping gives 157. Of the 118 additional applicable conditions, 78 have $U/D \leq 1.25$, 113 have $U/D \leq 2$, and 54 have $D = U$. The other seven need more than eight stable groups. Within the 39 common conditions, canonical grouping gives three tighter, 32 equal and four wider ratios. Filling the coarsest partition increases the count of ratios within 1.25 from 73 to 107 (Table 1).

A random-direction stress test uses eight Gaussian model-pair or digit-image menus at two thresholds: 16 dependent conditions at $E = 24, 48, k = 4, 6$. Only two admit any stable partition within eight groups; all tested prefix paths give zero full upper brackets at one second (Appendix L).

4.2 WHEN DOES JOINT INFORMATION CHANGE A POLICY COUNT?

The three-row/column family (11) scales from $E = 12$ to 2,048: bounds match (10) in 2.87–3.45 ms without batch enumeration. This illustrates compression, not prevalence.

On all small instances, Strong+ combines aggregate counting, exact/capped pairwise packing, single-type bounds at $G = 1, 2, 4, 8$, and mandatory-column preprocessing with residual counting. Clique search closes 163/164 conditions; the capped case remains included and cannot support a unique win against optimal packing. Fixed G4 adds zero prescribed-budget rejections, G8 adds three, and full refinement adds eleven. Applicability alone need not change an integer decision (Appendix K).

Exact residual-coverage greedy (Kurtz, 2026) supplies actual integer libraries; all 492 runs on 164 inputs pass coverage checks. Pairing the same greedy upper with Strong+ proves 103 exact policy counts. G8 adds none, although it narrows ten intervals; full refinement adds eleven exact counts. These differ from the eleven prescribed-budget rejections. Greedy price-dual changes one interval and neither conclusion (Table 2).

4.3 PARTIAL-DEMAND INFORMATION AND STRONGER SAMPLING CONTROLS

On the 30 larger-instance conditions ($E = 24, 48, 64$), the prefix-only LP has no budget advantage over 4,096-attempt sampled packing. Conservative counts at $Q = 256$ add twelve condition–budget decisions beyond the best of all three packing seeds, but ten are already explained by same-count

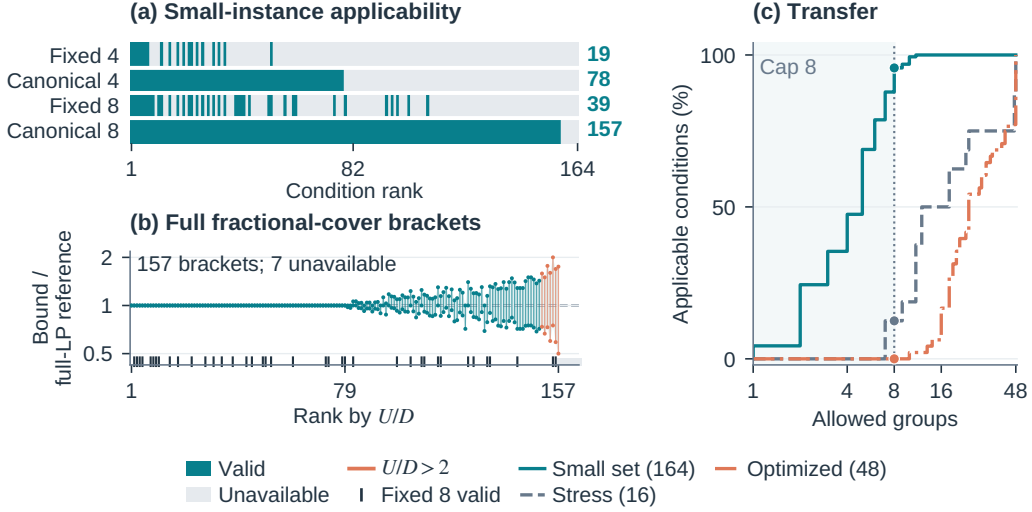


Figure 2: **Applicability, precision and a transfer boundary.** (a) All 164 conditions ordered by minimum stable-group count; teal is valid, gray unavailable; numbers give valid totals. (b) All 157 canonical eight-group brackets, normalized by a checked full-LP lower reference and ordered by U/D . Coral marks $U/D > 2$; dark bottom ticks identify fixed-G8 applicability. Seven unavailable conditions remain gray. (c) Exact cumulative minimum group counts for the small-instance set (teal), all 16 stress conditions (slate), and all 48 optimized witnesses (coral); dots mark cap eight. These intervals are certificates, not confidence intervals.

Lower portfolio	Exact χ	Extra exact	Narrower	Width sum	Removed budgets
Strong+	103	0	0	182	0
Strong+ and G4	103	0	2	180	2
Strong+ and G8	103	0	10	170	12
Strong+ and full LP	114	11	44	116	66

Table 2: Integer intervals using the same verified remaining-coverage greedy library on all 164 small instances. Each interval is $[\lceil L \rceil, K_{\text{gr}}]$; width sum adds $K_{\text{gr}} - \lceil L \rceil$ across inputs. Extra exact counts, narrowed intervals, and removed unresolved integer budgets are paired against Strong+. Blue marks positive incremental counts, including interval improvements that do not establish an exact integer optimum.

aggregate or single-row bounds. Only two require the joint LP: one block witness at $M = 1024$ and one Digits witness at $M = 64$. Packing conversely supplies eight decisions the rounded method misses. Rounded G4 adds none beyond Strong+ on the 164 explicit inputs. These exploratory results cannot establish an equal-time advantage (Appendix M).

At $M = 1024$, a 512-batch pool is inadequate as the sole control. Exploratory stress-test comparisons with uncapped packing and same-group LP-free counting give two LP-only versus ten control-only decisions at one second. Appendix N documents execution corrections and all runs.

4.4 OPTIMIZED WITNESSES: DECISIONS DEPEND ON THE CLOCK

The optimized-witness protocol is fixed before generation. It uses the sensing-task generator and joint-risk oracle on 16 Gaussian/decay tasks from eight seed-shape streams, $E = 24, 48$, $k = 4, 6$. First, middle and last distinct logged witnesses per task give 48 dependent conditions. All eleven methods have three seconds including grouping, counting, LP and verification, with identical uncapped streaming-packing fallbacks. Three runs rotate method order and record five checkpoints.

At 100 ms, rounded G4 rejects 166/240 condition-budget queries ($M = 4, 16, 64, 256, 1024$). Against the union of same-count LP-free control and pure packing, seven decisions across four

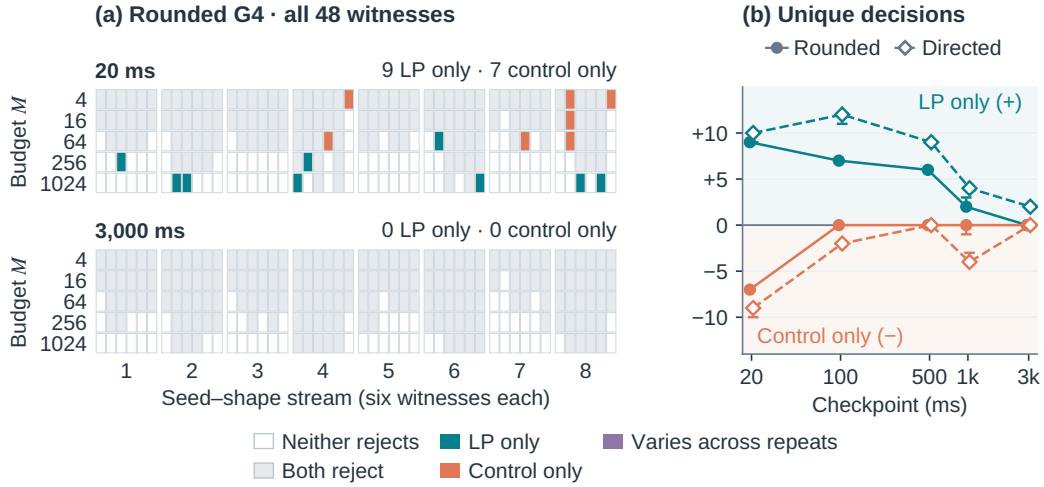


Figure 3: **Paired decisions on all optimized witnesses.** Controls combine same-information LP-free counting and pure uncapped packing. (a) All rounded-G4 condition-budget pairs: teal/coral mark LP-only/control-only rejection; gray both, white neither, purple disagreement across three run-time repetitions. Each seed-shape stream contributes six dependent witnesses. (b) Positive/negative values count LP-only/control-only decisions. Points show medians, bars the full range across three runs (not confidence intervals); lines join measured checkpoints.

streams are LP-only, with no converse decisions, in every repetition. At 500 ms the counts are six versus zero; at three seconds both paths reject the same 181 queries. Directed refinement retains two LP-only decisions at three seconds. Partial counts help at intermediate clocks, without establishing an eventual rounded-LP advantage (Figure 3).

Every method returns zero full-family upper brackets. Exact diagnostics prove that all 48 inputs need more than eight stable groups. A post-hoc ablation rejects that cap after eight witnessed necessary cuts in median .148 ms, without completing the partition. This makes refusal cheap, while leaving the substantive applicability boundary unchanged (Appendix O).

Controlled scaling. A prespecified 96-condition sweep from 16 weight streams varies $E = 12$ –96, $k = 2, 4, 6$ and group caps 4/8/16. Canonical methods give 4/16/30 full brackets at one second; fixed G4/G8 give none. All 24 exact cap-16 brackets are singleton refinements at $E = 12$. Rounded G4 retains two LP-only versus one control-only decision, with no LP-only gain at $E = 96$ (Appendix P). Direct sparse-dual optimization remains fastest in the separate 120-condition search study (Appendix Q).

5 CONCLUSION

Composition controls capacity beyond matched coverage statistics. Stable grouping preserves compatible demand; robust columns bound completion aggregation loss. Joint prices can exceed optimal packing by a growing factor. On optimized witnesses, partial counts add intermediate-clock decisions; rounded LP and its controls agree at three seconds.

Limitations. No deployed consumer is shown to require this representation; diagonal witnesses cost $O(k)$. Canonical grouping is combinatorial in k , and few groups need not give tight brackets. Eight groups provide limited compression on small instances and no extra exact integer counts over Strong+ with greedy. All 48 optimized witnesses need more than eight stable groups. Conditions are dependent; fractional uppers are not integer libraries, and dual query distributions need not match deployment. Unstable demand requires additional boundary coverage for full brackets.

REPRODUCIBILITY STATEMENT

The uploaded supplementary zip file with accompanying code package provides CPU replay entry-points and figure inputs; response-level replay also requires the separately retained frozen scientific bundles. The entrypoints reconstruct stored-output scores and statistics; they do not recreate provider inference.

AI USE STATEMENT

Generative AI assisted literature search and synthesis, implementation of the stimulus generators and analysis code, and manuscript proofreading.

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SUPPLEMENTARY MATERIAL

Finite Adaptability of Counterexample Libraries

The following appendices contain the proofs and additional experimental details. References appear immediately before this supplement. All cross-references and citations link within this PDF; theorem, equation, figure and table numbering continues from the main paper.

A ROBUST-DESIGN ORIGIN OF THE FIXED-PAIR PROBLEM

Let $P \subset \mathbb{R}^p$ be a nonempty compact parameter set, $q(\theta) = b^\top \theta$ a scalar target, and $g(\theta) = A\theta \in \mathbb{R}^E$ the full measurement prediction. Affine constants cancel in model differences. Write $K = P - P$. A batch $B \subseteq [E]$ with $|B| = k$ is chosen before observing $y_B = g_B(\theta) + \xi_B$. Throughout, ξ is one full-menu error vector in a fixed compact set Ξ ; the batch uses its coordinate projection $\pi_B \Xi$. The convex algorithms below additionally require P to be convex. Finite dictionaries satisfy the compatibility identities but not that extra assumption.

The largest possible posterior target diameter is

$$R(B) = \max_{d \in K} \{b^\top d : A_B d \in \pi_B(\Xi - \Xi)\}, \quad R^* = \min_{|B|=k} R(B). \quad (12)$$

The formula follows because two model predictions have a common admissible observation exactly when their difference belongs to the difference of the projected error sets. Symmetry of K and $\Xi - \Xi$ permits a signed target maximum. No statistical coverage claim is needed for this deterministic identity; such a claim would require a separately justified error set and model class.

For a model difference $d \in K$, define $z = Ad$ and $\Delta = b^\top d \geq 0$. Its *projected certificate* supplies the lower bound

$$L_d(B) = \Delta \mathbf{1}\{z_B \in \pi_B(\Xi - \Xi)\} \leq R(B). \quad (13)$$

Thus $R(B) \leq \tau < \Delta$ requires $z_B \notin \pi_B(\Xi - \Xi)$. For rectangular Ξ , membership factorizes coordinatewise and the requirement reduces to hitting the ordinary singleton separator set. It does not factorize for a general joint set.

To define the lifted alternative precisely, take one $h \in \Xi - \Xi$. The lifted model pair has augmented prediction difference $z - h$ and cancellation set $C(d, h) = \{e : z_e = h_e\}$. This pair supplies the lower bound Δ only for batches $B \subseteq C(d, h)$. The corresponding singleton separator mask is $[E] \setminus C(d, h)$. Figure 1 shows why this valid certificate can retain less information than (13).

For ellipsoidal error $\Xi = \{\xi : \xi^\top \Sigma^{-1} \xi \leq c^2\}$, put $T = 4c^2$ and

$$Q_z(B) = z_B^\top \Sigma_{BB}^{-1} z_B, \quad Q_z(\emptyset) = 0. \quad (14)$$

Then compatibility is exactly $Q_z(B) \leq T$. The minimum-energy full-menu completion is $h = \Sigma_{:B} \Sigma_{BB}^{-1} z_B$. Thus inverse and restriction cannot be interchanged. The shape matrix parameterizes a deterministic error set; no statistical coverage follows without a separately justified model and radius. For $T > 0$, the normalized full-menu radius used below is $\max_{d \in K} \sqrt{Q_{Ad}([E])}/T$.

B THE UNION CRITERION AND CLASSICAL FRACTIONAL COVERING

Proposition 1 (Budget-specific lossless compilation). *For an arbitrary positive-definite shape Σ , define $\mathcal{F}_{z,k} = \{B : |B| = k, Q_z(B) \leq T\}$, where $Q_z(B) = z_B^\top \Sigma_{BB}^{-1} z_B$, and let $J_{z,k} = \bigcup_{B \in \mathcal{F}_{z,k}} B$. If $\mathcal{F}_{z,k}$ is empty, its cover number is zero. Otherwise one completion covers it exactly when $Q_z(J_{z,k}) \leq T$. For diagonal shapes and $1 \leq k \leq E$, a coordinate e belongs to $J_{z,k}$ exactly when*

$$w_e + \text{the sum of the } k-1 \text{ smallest weights outside } e \leq T.$$

Consequently the whole union and its losslessness test are computable by sorting and prefix sums in $O(E \log E)$ arithmetic/comparison operations. For $k = 0$ and $T \geq 0$, there is one empty batch and one completion suffices.

Proof. A completion covering every compatible k -batch must match z on their union, so that union must be compatible. Conversely, a compatible union has the minimum-energy completion $h = \Sigma_{:,J} \Sigma_{J,J}^{-1} z_J$ (take $h = 0$ when J is empty), which covers every member. In the diagonal case, the cheapest k -set containing e consists of e and the $k - 1$ cheapest other coordinates, proving the test. To implement all tests after sorting $w_{(1)} \leq \dots \leq w_{(E)}$, let $S_j = \sum_{i=1}^j w_{(i)}$. The cheapest containing-set cost for rank $i \leq k$ is S_k , and for $i > k$ it is $w_{(i)} + S_{k-1}$. Ties do not change either containing-set minimum. The empty-family and empty-batch conclusions follow directly from the definition of a cover. $\square$

B.1 EXACT FRACTIONAL REDUCTION AND FINITE INTEGER BOUNDS

Let $U = \{h \in \mathbb{R}^E : h^\top \Sigma^{-1} h \leq T\}$, where $T \geq 0$ and $\Sigma = \text{diag}(\sigma_e)$ with $\sigma_e > 0$. Fix $z \in \mathbb{R}^E$. Partition the coordinates into G groups of sizes $E_g \geq 1$ such that $z_e^2/\sigma_e = w_g \geq 0$ within group g . Write $E = \sum_g E_g$, fix $0 \leq k \leq E$, and define

$$\begin{aligned} \mathcal{B} &= \{b \in \mathbb{Z}_+^G : b \leq (E_g)_g, \sum_g b_g = k, \sum_g w_g b_g \leq T\}, \\ \mathcal{C} &= \{c \in \mathbb{Z}_+^G : c \leq (E_g)_g, \sum_g w_g c_g \leq T\}. \end{aligned}$$

A batch is compatible if it is a k -set of type $b \in \mathcal{B}$. Let $\mathcal{U}$ be the family of these batches and $N = |\mathcal{U}| = \sum_{b \in \mathcal{B}} N_b$, where $N_b = \prod_g \binom{E_g}{b_g}$. Assume $\mathcal{B} \neq \emptyset$ until stated otherwise. Let M be the minimum number of admissible completions whose cancellation sets cover $\mathcal{U}$.

A coordinate set C can be canceled by a completion exactly when $\sum_{e \in C} z_e^2/\sigma_e \leq T$. Every actual cancellation set satisfies this inequality. Conversely, setting $h_e = z_e$ on C and $h_e = 0$ elsewhere gives an admissible completion. Its cancellation set may additionally contain zero-prediction coordinates; this only increases coverage. Thus both integral and fractional completion-cover optima equal the corresponding set-cover optima using *all* feasible coordinate sets C . This observation includes $w_g = 0$ and does not assume every feasible set is an exact cancellation set.

Define, with $\binom{c_g}{b_g} = 0$ for $c_g < b_g$,

$$a_{bc} = \prod_{g=1}^G \frac{\binom{c_g}{b_g}}{\binom{E_g}{b_g}}, \quad p_b = \max_{c \in \mathcal{C}} a_{bc}, \quad q = \max_{b \in \mathcal{B}} p_b^{-1}, \quad L = |\mathcal{B}|.$$

Here $p_b > 0$ because $c = b$ is feasible, and $q \geq 1$.

Theorem 4 (Exact orbit-reduced fractional cover). *The fractional completion-cover optimum equals*

$$\tau^* = \min_{x_c \geq 0} \left\{ \sum_{c \in \mathcal{C}} x_c : \sum_{c \in \mathcal{C}} a_{bc} x_c \geq 1 \text{ for every } b \in \mathcal{B} \right\}. \quad (15)$$

The variables x_c are total fractional mass on each completion orbit; they are not integer counts of representative completions. Moreover,

$$q \leq \tau^* \leq \sum_{b \in \mathcal{B}} p_b^{-1} \leq Lq, \quad \tau^* \leq M \leq \min\{N, \lceil (1 + \log N) \tau^* \rceil\}. \quad (16)$$

All logarithms are natural.

Proof. Start with one fractional variable y_C for each feasible coordinate set, and constraints $\sum_{C \supseteq B} y_C \geq 1$ for each $B \in \mathcal{U}$. Average a feasible solution over independent coordinate permutations within each group. Feasibility and objective are unchanged, and the average has a common weight within each completion-type orbit. If that orbit has total mass x_c , its contribution to a fixed batch of type b is

$$x_c \prod_g \frac{\binom{E_g - b_g}{c_g - b_g}}{\binom{E_g}{c_g}} = x_c a_{bc}.$$

Checked object	Required condition	Consequence
Dual value L	Exact or conservative demand counts; all original completion constraints	$L \leq \tau^* \leq \chi$
Optimistic primal cost V	Demand stability and contraction factor ρ	$\tau^* \leq V/\rho$
Robust primal cost W	Demand stability	$\tau^* \leq W$
Concrete library of size K	Legal entries covering the actual family	$\chi \leq K$

Table 3: Certificate contracts. Feasibility suffices; a numerical optimum claim is unnecessary. $L > M$ rejects integer capacity M , but a fractional primal does not establish a concrete library. L/ρ is not an upper certificate.

The ratio is zero when containment is impossible. Conversely, distribute any feasible x_c uniformly over its $\prod_g \binom{E_g}{c_g}$ sets. The resulting y_C satisfies every original constraint with identical objective. This proves exact equality of the two fractional optima.

For each row b , feasibility implies $1 \leq \sum_c a_{bc}x_c \leq p_b \sum_c x_c$, giving the lower bound q . For an upper bound, for every b place mass p_b^{-1} on a maximizing type $c(b)$. Summing these masses gives a feasible solution: each row receives at least its own unit contribution. This proves the first chain.

Every integer cover is fractional, and choosing one completion per batch gives $M \leq N$. For the logarithmic upper bound, take an optimal original fractional solution y_C and draw $r = \lceil (1 + \log N)\tau^* \rceil$ sets independently from $\Pr(C) = y_C/\tau^*$. Each batch is covered in one draw with probability at least $1/\tau^*$. Its probability of being uncovered after r draws is at most $\exp(-r/\tau^*)$. The expected number of uncovered batches is at most $N \exp(-r/\tau^*) \leq e^{-1} < 1$. An outcome therefore covers all batches, since the number left uncovered is a nonnegative integer. Removing repeated draws cannot increase the number of completions. This is an existence bound, not an efficient algorithm for finding or verifying the resulting cover. $\square$

C CERTIFIED COMPLETION PROFILES FOR ARBITRARY DIAGONAL WEIGHTS

This section proves the profile hierarchy and its certificate variants.

C.1 A PROFILE BASED ON COUNTS AND WITHIN-GROUP ORDER STATISTICS

Let $w_e \geq 0$, $T \geq 0$, $1 \leq k \leq E$, and

$$\mathcal{F} = \{B \subseteq [E] : |B| = k, \sum_{e \in B} w_e \leq T\}, \quad \mathcal{C} = \{C \subseteq [E] : \sum_{e \in C} w_e \leq T\}.$$

Assume $\mathcal{F} \neq \emptyset$. All type vectors have integer entries with $0 \leq b_g, c_g \leq E_g$ wherever defined. Partition coordinates into any groups $I_1, \dots, I_G$ of sizes E_g . Write $t(S)_g = |S \cap I_g|$ and set $n_b = |\{B \in \mathcal{F} : t(B) = b\}|$. Sort the weights within group g , and let $s_g^-(j)$ and $s_g^+(j)$ be the sums of its j smallest and j largest weights, respectively; both are zero at $j = 0$. Define

$$\mathcal{T}^+ = \{c : 0 \leq c_g \leq E_g, \sum_g s_g^-(c_g) \leq T\}, \quad K_{bc} = \prod_g \binom{c_g}{b_g}.$$

Thus $\mathcal{T}^+$ contains exactly the types of cancelable coordinate sets. It can be enumerated from prefix sums, without enumerating all coordinate subsets. A type with $c_g < b_g$ has $K_{bc} = 0$. If a zero-prediction coordinate is matched incidentally as well, the actual cancellation set may be larger. This causes no change in the cover optimum: every listed feasible set is realizable as a subset of an actual cancellation set, so its column is dominated by a realizable one. Actual cancellation sets themselves are listed.

Theorem 5 (Counted-type lower bound and refinement). *For an arbitrary partition and arbitrary nonnegative weights, define*

$$D_{\mathcal{P}} = \min_{x \geq 0} \left\{ \sum_{c \in \mathcal{T}^+} x_c : \sum_c K_{bc} x_c \geq n_b \text{ for every } b : n_b > 0 \right\} \quad (17)$$

$$= \max_{\lambda \geq 0} \left\{ \sum_b n_b \lambda_b : \sum_b K_{bc} \lambda_b \leq 1 \text{ for every } c \in \mathcal{T}^+ \right\}. \quad (18)$$

If τ^ is the fractional cover number of the actual compatible family and χ its minimum number of concrete completions, then $D_{\mathcal{P}} \leq \tau^* \leq \chi$. Refining the partition cannot decrease $D_{\mathcal{P}}$. A one-group partition gives $D_{\mathcal{P}} = |\mathcal{F}| / \binom{m_*}{k}$, where m_* is the largest feasible cancellation cardinality. Singleton groups give exactly $D_{\mathcal{P}} = \tau^*$. If weights are constant within every group, the program is exactly the orbit fractional cover after rescaling its rows by $n_b = \prod_g \binom{E_g}{b_g}$.*

Proof. A type c has a legal realization precisely when its cheapest realization has energy at most T , proving the prefix-sum description. Every legal completion C of type c contains exactly K_{bc} size- k subsets of type b . All these subsets are compatible by nonnegativity of the weights. Sum the individual batch-cover inequalities within each type and aggregate completion variables by their types. This maps every feasible actual fractional cover to a feasible solution of (17) with the same cost. Finite LP duality gives (18); feasibility and boundedness follow by taking one completion per compatible batch and by $n_b > 0$.

For refinement, lift any feasible coarse dual vector by assigning each refined batch type the multiplier of its coarse type. Counts add to the coarse n_b , so the objective is unchanged. Every legal refined completion type has a legal coarse type. For a legal realization of that refined type, summing the numbers of its subsets of each refined batch type gives the corresponding coarse count K_{bc} (equivalently, Vandermonde’s identity within each split group). Types with zero compatible count contribute zero: a subset of a legal completion cannot be incompatible. The lifted dual constraints are therefore precisely the coarse constraints on these columns, and remain feasible. This proves monotonicity. The one-group and singleton cases follow directly. Under equal weights all realizations of a type are feasible or infeasible together, and the symmetrization theorem applies. $\square$

The normalized main-text dual uses $y_b = n_b \lambda_b$. Singleton refinement gives a one-row bound of one but retains the full fractional cover in the joint LP. There are at most $(E+1)^G$ completion types; counting compatible requests can still be expensive when k grows. The dual omission guarantee (8) follows by assigning mass $\lambda_{t(B)}/L$ to each compatible request, where $L = \sum_b n_b \lambda_b$. The more general adaptive-cache result in Appendix J proves the same bound and its conditional version. This distribution is certificate-defined, not a workload or an overall robust-risk guarantee.

An approximate numerical dual need not be trusted unmodified. For any nonnegative candidate $\hat{\lambda}$, compute an upper bound $r \geq \max_{c \in \mathcal{T}^+} \sum_b K_{bc} \hat{\lambda}_b$. For $r > 0$, $\hat{\lambda}/r$ is feasible and certifies $\sum_b n_b \hat{\lambda}_b / r$. Enumerated columns allow this check with exact rational arithmetic. A restricted-column LP alone provides no certified lower bound until the omitted columns have been priced or otherwise bounded.

C.2 CERTIFICATES WITHOUT COUNTING PARTIALLY COMPATIBLE TYPES

Define

$$\mathcal{D}^- = \{b : \sum_g b_g = k, \sum_g s_g^+(b_g) \leq T\}, \quad \mathcal{T}^- = \{c : \sum_g s_g^+(c_g) \leq T\}.$$

All realizations of a type in $\mathcal{D}^-$ are compatible and all realizations of a type in $\mathcal{T}^-$ are legal completions. Let $N_b = \prod_g \binom{E_g}{b_g}$ and $a_{bc} = K_{bc}/N_b$. Let $\ell_{\mathcal{P}}$ and $U_{\mathcal{P}}$ be the orbit LP values covering the demand types $\mathcal{D}^-$ using columns $\mathcal{T}^+$ and $\mathcal{T}^-$, respectively. Both use constraints $\sum_c a_{bc} x_c \geq 1$. Put $N^- = \sum_{b \in \mathcal{D}^-} N_b$, and let R be the number of truly compatible batches outside these complete types. Then

$$\ell_{\mathcal{P}} \leq \chi \leq R + \min\{N^-, \lceil (1 + \log N^-) U_{\mathcal{P}} \rceil\}, \quad (19)$$

where the second summand is zero when $N^- = 0$. The lower bound holds by retaining only certified demands and permitting a superset of legal completions. For the upper bound, the symmetric abstract problem has fractional optimum $U_{\mathcal{P}}$; every permitted completion is legal for the original weights. Standard fractional-cover rounding covers these N^- batches with the displayed number of completions, and one additional completion per residual batch covers the rest. If R is not known, it can be bounded by

$$\sum_{b: \sum b_g = k, \sum_g s_g^-(b_g) \leq T < \sum_g s_g^+(b_g)} N_b.$$

Replacing prefix sums by $j \min_{e \in I_g} w_e$ and $j \max_{e \in I_g} w_e$ weakens these certificates but preserves validity. It is not valid simply to round weights up and down and compare the two ordinary cover numbers: rounding changes both the demand family and the available completions.

Corollary 1 (Exact reduction with distinct weights and a margin). *If every completion type c satisfies either $\sum_g s_g^+(c_g) \leq T$ or $\sum_g s_g^-(c_g) > T$, then feasibility is constant on every type, $\mathcal{T}^- = \mathcal{T}^+$, $R = 0$, and the counted LP is the exact fractional cover. All grouped-profile and logarithmic-rounding conclusions therefore hold even when every coordinate weight is different.*

Proof. The stated alternatives exclude every type with both legal and illegal realizations, making the feasible downset invariant under within-group permutations. Symmetrization, not numerical equality of the weights, is the property required for the exact orbit proof. $\square$

C.3 A QUANTITATIVE CERTIFICATE FOR INFORMATION RETAINED BY GROUPING

For a type v , write $L(v) = \sum_g s_g^-(v_g)$ and $U(v) = \sum_g s_g^+(v_g)$. The following condition concerns only demand types, and is weaker than requiring the complete feasible downset to be invariant:

$$|b| = k, L(b) \leq T \implies U(b) \leq T. \quad (\text{DS})$$

It is an observable test using the input weights, not an assumption that unequal weights may be treated as equal. Under (DS), every compatible batch type is fully compatible, so $n_b = N_b$ and the input-only and counted lower LPs coincide. Completion types may still have both feasible and infeasible realizations.

Let $\mathcal{C}$ contain the componentwise maximal members of $\mathcal{T}^+$ with cardinality at least k ; these suffice in the lower LP because all other useful columns are dominated. For $c \in \mathcal{C}$, let $m_c = |c|$ and form the multiset V_c consisting of the c_g largest weights from each group. Let $H_c(j)$ be the sum of the j largest entries in V_c , and put

$$r_c = \max\{j \leq m_c : H_c(j) \leq T\}, \quad \rho_c = \frac{\binom{r_c}{k}}{\binom{m_c}{k}}, \quad \rho = \min_{c \in \mathcal{C}} \rho_c. \quad (20)$$

Here r_c is the cardinality that is safe under every realization and every choice of retained coordinates of type c .

Theorem 6 (Prefix contraction controls fractional information loss). *Suppose the weights are non-negative, the compatible family is nonempty, and (DS) holds. Then $r_c \geq k$ for every $c \in \mathcal{C}$, and $\rho > 0$. Let $D_{\mathcal{P}}$ be the counted lower LP, τ^* the actual fractional cover number, and $U_{\mathcal{P}}$ the orbit fractional cover using only the robust columns $\mathcal{T}^-$. Then*

$$D_{\mathcal{P}} \leq \tau^* \leq U_{\mathcal{P}} \leq \frac{D_{\mathcal{P}}}{\rho}. \quad (21)$$

Consequently, with $N = \sum_{b \in \mathcal{D}^-} N_b$,

$$\chi \leq \min \left\{ N, \left\lceil (1 + \log N) \frac{D_{\mathcal{P}}}{\rho} \right\rceil \right\}. \quad (22)$$

If $\max_{c \in \mathcal{C}} k(m_c - r_c)/m_c \leq \varepsilon < 1$, then $\rho \geq 1 - \varepsilon$, giving $\tau^* \leq D_{\mathcal{P}}/(1 - \varepsilon)$. The demand test and ρ are computable without enumerating compatible batches. Enumerating at most $P = \prod_g (E_g + 1)$ types and sorting the worst-realization multisets takes $O(PE \log E)$ arithmetic operations; for fixed G their exact rational verification has polynomial bit cost. This does not promise a polynomial-size exported concrete library.

Proof. Fix $c \in \mathcal{C}$. Every type $b \leq c$ with $|b| = k$ is min-prefix feasible, since it occurs inside a cheapest realization of c . Condition (DS) makes every such type fully feasible. The k largest entries of V_c have one of these types, hence $H_c(k) \leq T$ and $r_c \geq k$.

Choose uniformly r_c labels out of m_c abstract labels, of which c_g belong to group g . Their retained type d has probability

$$p_c(d) = \frac{\prod_g \binom{c_g}{d_g}}{\binom{m_c}{r_c}} \quad (d \leq c, |d| = r_c). \quad (23)$$

Every realization of d has weight at most $H_c(r_c) \leq T$: its maximum weight is obtained by choosing the d_g largest coordinates in each group, all among V_c . Thus $d \in \mathcal{T}^-$. Each one of the K_{bc} abstract k -subsets of type b is retained with probability $\binom{r_c}{k} / \binom{m_c}{k}$, so

$$\sum_d p_c(d) K_{bd} = \rho_c K_{bc} \quad \text{for every demand type } b. \quad (24)$$

Take any feasible solution x of the lower orbit LP, normalized with $a_{bc} = K_{bc}/N_b$. Replace its mass x_c by mass $x_c p_c(d)/\rho_c$ on robust type d . Equation (24) preserves every row's coverage, and the new cost is $\sum_c x_c/\rho_c \leq \sum_c x_c/\rho$. This proves $U_{\mathcal{P}} \leq D_{\mathcal{P}}/\rho$. The robust-type system is symmetric and all its concrete columns are legal. Distributing each robust type's mass uniformly over its realizations therefore gives an actual fractional cover, proving $\tau^* \leq U_{\mathcal{P}}$. The reverse lower bound is the counted aggregation bound. With log denoting the natural logarithm, classical fractional set-cover rounding gives (22).

For the last inequality, view the contraction as deleting $m_c - r_c$ uniformly selected labels. Each fixed label is deleted with probability $(m_c - r_c)/m_c$. A union bound on the k labels of a fixed batch gives $\rho_c \geq 1 - k(m_c - r_c)/m_c$. Finally all type counts, prefix comparisons, and binomial coefficients have polynomial bit length for fixed G . $\square$

Algorithmic consequence. For rational weights and threshold, fixed G , and a fixed $\rho_0 > 0$, demand-stable instances with $\rho \geq \rho_0$ admit a polynomial-time $1/\rho_0$ approximation of the actual fractional cover number. Indeed, the two type LPs have at most P rows and columns and polynomial-length rational coefficients. Polynomial-time rational LP algorithms compute their optima; the theorem bounds their ratio. This is a consequence of the structural guarantee and standard LP complexity, not a claim about a particular solver's timing. The polynomial depends on fixed G , and an explicit concrete library may still be exponential.

Sharper computable bounds. The proof gives $U_{\mathcal{P}} \leq \sum_c x_c/\rho_c$ for each feasible lower-LP solution x . Thus the auxiliary LP with costs $1/\rho_c$, instead of unit costs, can improve the uniform worst-column factor. Alternatively, solving the robust LP itself certifies the sharper interval $[D_{\mathcal{P}}, U_{\mathcal{P}}]$. These are input-only LPs under demand stability.

If $U(c) \leq T$ for every feasible completion type, then $r_c = m_c$ and $\rho = 1$, recovering exact margin symmetry. More generally, when all weights are at least $a > 0$, deleting $q = \lceil (U(c) - T)_+ / a \rceil$ coordinates is sufficient whenever $q \leq m_c - k$: the sum of the q smallest entries in V_c is at least qa . This relates the retention ratio to an observable threshold excess, but the exact order-statistic definition also handles zero weights.

Partially compatible demand types must be accounted for. If (DS) fails, the theorem's full-family multiplicative guarantee is not asserted. In fact, for any partially compatible type b , every robust column $d \in \mathcal{T}^-$ satisfies $K_{bd} = 0$. Otherwise $b \leq d$ would make every realization of b extendible to a realization of d , implying $U(b) \leq U(d) \leq T$, a contradiction. Robust columns cannot cover even the compatible members of a partially compatible type.

There is still an explicit residual guarantee. Set $\mathcal{S} = \mathcal{D}^-$, let $\ell_{\mathcal{P}}$ be its lower orbit LP, and let R count truly compatible batches in types outside $\mathcal{S}$. For every relevant column c , a supplied probability distribution p_c over robust subtypes $d \leq c$ satisfying

$$\sum_d p_c(d) K_{bd} \geq \rho K_{bc} \quad \text{for all } b \in \mathcal{S} \quad (25)$$

for a common $\rho > 0$ certifies

$$\ell_{\mathcal{P}} \leq \tau^* \leq U_{\mathcal{P}} + R \leq \ell_{\mathcal{P}}/\rho + R, \quad \chi \leq R + \min\{N^-, \lceil (1 + \log N^-)\ell_{\mathcal{P}}/\rho \rceil\}. \quad (26)$$

Here the second summand is zero if $N^- = 0$; no positive ρ need be specified in that empty-support case. The integer bound covers each residual compatible batch by itself. Without actual counts, the input weights supply the sound upper bound

$$R \leq R^+ = \sum_{|b|=k: L(b) \leq T < U(b)} N_b. \quad (27)$$

The supplied rational kernels, their robust support, and all inequalities in (25) can be checked by type enumeration in polynomial bit complexity for fixed G . Positive kernels always exist when $\mathcal{S}$ is nonempty: for a column covering certified demands, let $M_c = \sum_{b \in \mathcal{S}} K_{bc} > 0$ and put $p_c(d = b) = K_{bc}/M_c$. Since all these types have size k , this attains $\rho_c = 1/M_c$. This fallback can be very loose; it does not remove the significance of partial demand types or the residual cost.

Scope and inherited ingredients. Corollary 1 gives the exact-margin endpoint. The additional statement is the quantified loss certificate under only demand stability, derived from worst-realization prefix sums and hypergeometric deletion. Symmetrization, linear-program column domination, and logarithmic set-cover rounding are classical ingredients. This is a sufficient condition for small groups to retain fractional-cover information, not a characterization, a novel rounding principle, a packing dominance theorem, or evidence of practical prevalence.

Proposition 2 (Order-tight retention loss for pair demands). *For each integer $n \geq 3$, take one group containing n distinct light weights in $[1 - 1/(4n), 1]$ and n distinct heavy weights in $[n/2 - 1/4, n/2]$, with $k = 2$ and $T = n$. All coordinate weights are distinct and all pairs are compatible. Then*

$$D_{\mathcal{P}} = \frac{\binom{2n}{2}}{\binom{n}{2}} \leq 5, \quad \rho = \frac{1}{\binom{n}{2}}, \quad U_{\mathcal{P}} = \binom{2n}{2} = \frac{D_{\mathcal{P}}}{\rho}, \quad \tau^* \geq \binom{n}{2} \geq \frac{D_{\mathcal{P}}}{5\rho}.$$

Thus the $1/\rho$ dependence cannot be improved in order even with one group, distinct weights, stable demand types, and fixed batch size two.

Proof. Every heavy weight exceeds every light weight. The two largest weights sum to at most n , so all pairs are feasible and demand stability holds. All n light coordinates fit in one completion. Every set of n coordinates containing a heavy, however, weighs at least

$$\left(\frac{n}{2} - \frac{1}{4}\right) + (n-1) \left(1 - \frac{1}{4n}\right) = \frac{3n}{2} - \frac{3}{2} + \frac{1}{4n} > n.$$

Any larger set contains such an infeasible n -subset. Hence the maximum feasible cardinality is exactly n , giving the stated one-group lower LP value. Algebraically this value is $4 + 2/(n-1) \leq 5$.

Every pair is robustly feasible, but the three largest weights weigh at least $3n/2 - 3/4 > n$. Thus robust completions have size at most two and their fractional optimum is the number of pair demands, $\binom{2n}{2}$. For the maximal min-prefix type $c = n$, the worst realization consists of all n heavies; its retained safe cardinality is $r_c = 2$. Therefore $\rho = \binom{2}{2}/\binom{n}{2}$.

Finally, no heavy-heavy pair can be extended by any coordinate: adding even the minimum permitted light weight gives total weight at least $n - 1/2 + 1 - 1/(4n) = n + 1/2 - 1/(4n) > n$. Consequently each of the $\binom{n}{2}$ heavy-heavy demands has only its own pair as a legal completion containing it. Each requires one unit of fractional cover mass, proving $\tau^* \geq \binom{n}{2}$ and the final bound. $\square$

D THE MINIMUM DEMAND-STABLE PARTITION

For a group-cap decision, $G_{\max}$ distinct necessary adjacent cuts are a certificate that at least $G_{\max} + 1$ groups are needed. The enumeration may stop immediately at that point, retaining the corresponding contexts as independently checkable witnesses. Such a return proves infeasibility of the cap, not the exact minimum. If it does not find this obstruction, accepting a stable partition still requires completing the exact test (or another valid full certificate).

Write $\mathcal{F} = \{B \subseteq [E] : |B| = k, w(B) \leq T\}$. A partition is demand-stable precisely when membership in $\mathcal{F}$ depends only on group cardinalities: the minimum- and maximum-prefix tests determine whether a type is empty, full, or partially compatible. The transposition definition of hypergraph twins is classical (Babai, 2017, Definition 2.5.6 and Observation 2.5.8). We use it to compute a grouping from an implicit threshold family; we make no claim to introduce twin equivalence or symmetry aggregation.

D.1 CHARACTERIZATION AND DIRECT COMPUTATION

Coordinates i, j are demand twins if and only if

$$R \cup \{i\} \in \mathcal{F} \iff R \cup \{j\} \in \mathcal{F} \quad \text{for every } R \in \binom{[E] \setminus \{i, j\}}{k-1}.$$

Sets containing both or neither coordinate are unchanged by the transposition, so this condition is equivalent to its preserving $\mathcal{F}$.

Proof of Theorem 3. Reflexivity and symmetry are immediate. If (ij) and (jh) preserve $\mathcal{F}$, their conjugate $(ij)(jh)(ij) = (ih)$ also preserves it. Thus the twin relation is an equivalence relation. Every within-group transposition must preserve a demand-stable family, so each stable group lies within a twin class. Conversely, the transpositions within a twin class generate all its permutations. Any two sets with the same class cardinalities are related by these permutations. The class partition and all its refinements are therefore stable, proving unique coarseness.

For a short proof of contiguity, let $d_i = |\{B \in \mathcal{F} : i \in B\}|$. If $w_i \leq w_j$, the contributions of sets containing both i, j to $d_i - d_j$ cancel, and hence

$$d_i - d_j = \sum_{R \in \binom{[E] \setminus \{i, j\}}{k-1}} (\mathbf{1}\{w(R) + w_i \leq T\} - \mathbf{1}\{w(R) + w_j \leq T\}).$$

Every summand is nonnegative. Consequently $d_i \geq d_j$, with equality if and only if i, j are twins. Equal-degree classes are contiguous in sorted weight order. For adjacent positions, a nonzero summand is exactly the witness interval (9).

Enumerate every $(k-1)$ -set R once and put $t = T - w(R)$. Let i be the last sorted position with $w_i \leq t$. If $1 \leq i < E$ and both $i, i+1$ are absent from R , save this witness for the cut. If either endpoint belongs to R , this context cannot witness that cut and must be discarded. Each retained witness is valid. Conversely, any witness of (9) gives this exact binary-search position and is retained when enumerated. Splitting precisely at the witnessed cuts therefore returns the twin classes. Equal weights cannot be separated by the witness interval. Sorting costs $O(E \log E)$; each context costs $O(k + \log E)$. Storing at most one context for each adjacent cut uses $O(Ek)$ auxiliary space, apart from the input and output. $\square$

The degree identity also gives an independent implementation: for each R , increment the eligible sorted prefix $\{i : w_i \leq T - w(R)\}$, then subtract the increment at each eligible coordinate already in R . Prefix difference arrays accumulate all degrees in the same asymptotic time. Each compatible k -set contributes once to the degree of each of its coordinates, through the unique context obtained by deleting that coordinate.

Degenerate layers and exact comparisons. For $E = 0$ the empty partition has zero groups. For $E > 0$, an empty or complete size- k demand family has one twin class. This includes $k = 0$, $k > E$, and $k = E$ as all-or-none layers. Repeated and zero weights are allowed. Rational inputs can be compared exactly after common denominator scaling. The operation bound is arithmetic, not independent of input bit length, and is polynomial only for fixed k .

D.2 APPLICABILITY, REFINEMENT, AND LIMITS

Corollary 2 (Exact applicability at a group cap). *If $G_{\max} < G_{\text{DS}}$, no partition into at most $G_{\max}$ groups has demand stability. In particular, no full-family upper certificate made solely from robust column orbits of such a partition is possible. If $G_{\max} \geq G_{\text{DS}}$, the canonical partition constructs a demand-stable grouping within that cap.*

Proof. The first and last statements follow from unique coarseness. For the robust-orbit statement, an unstable demand type b contains both a compatible request and an incompatible request. A robust column type c covering either request must satisfy $c \geq b$. Every realization of b then extends to some realization of c . If all realizations of c were feasible, nonnegativity would make every realization of b feasible, a contradiction. Thus compatible requests of type b receive no coverage from any robust column orbit. $\square$

This obstruction concerns the stated representation. It does not rule out explicit boundary repair, nonsymmetric columns, or other upper certificates. In particular the residual bounds of Appendix C.3 remain available.

Starting from the canonical partition, split a largest nonsingleton block at its median; choose the leftmost block when sizes tie. Stop at $\min\{G_{\max}, E\}$ groups, or earlier at a prescribed fractional precision. Each partition refines its predecessor, so stability persists and the partition hierarchy gives $D_{\mathcal{P}} \leq D_{\mathcal{P}'}$. Moreover $U_{\mathcal{P}'} \leq U_{\mathcal{P}}$: a uniform orbit of robust coarse columns is a mixture of robust refined orbits, with unchanged total mass and coverage. Thus the old robust fractional cover remains expressible after refinement. Neither monotonicity implies domination of a different, non-nested partition using the same number of groups.

When $D_{\mathcal{P}} > 0$, the test $U_{\mathcal{P}}/D_{\mathcal{P}} \leq 1 + \varepsilon$ certifies a relative bracket for the true fractional optimum. If the purpose is to reject an integer capacity M , the valid test is $D_{\mathcal{P}} > M$. A robust fractional upper at most M means that this fractional relaxation cannot reject the capacity; it does not construct an integer cover. Integer search or an unresolved outcome may still be required.

Proposition 3 (Arbitrarily narrow weights can require almost full refinement). *For even $E = 2n \geq 4$, take $k = 2$, $w_i = i$ and $T = E + 3/2$. The only nonsingleton twin class is $\{n, n + 1\}$, and $G_{\text{DS}} = E - 1$. For every $\eta > 0$, an equivalent pair-feasibility instance has all weights in $[1, 1 + \eta]$.*

Proof. For adjacent position i , the only integer context weight in (9) is $w_j = E + 1 - i$. This is an allowed context unless $i = n$, when $j = i + 1$. Thus precisely that cut is absent. Under $w'_i = a + bw_i$, $b > 0$, and $T' = 2a + bT$, pair feasibility is unchanged. Choose $b = \eta/(E - 1)$ and $a = 1 - b$. $\square$

Hence small weight spread alone guarantees neither a small stable partition nor inexpensive exact preprocessing. A small G_{DS} likewise does not control the retention factor or the width of the fractional bracket.

Calculator contract. The implementation accepts nonnegative integer or rational weights, an exact threshold, and k . Exact completion returns the classes as original coordinate indices, G_{DS} , sorted positions of required cuts, and one excluding-endpoint context per cut. Optional context and wall-clock caps return an incomplete status and only a certified lower bound on G_{DS} ; the partially cut partition is not reported as stable. Thus a timeout cannot produce a full-family upper claim. An interruption after enough distinct cuts have already forced more than $G_{\max}$ groups can still certify inapplicability at that cap.

E A CERTIFICATE THAT DOES NOT ENUMERATE COMPATIBLE BATCHES

The counted hierarchy uses an existing batch pool. A different operating regime is available when the pool is too large to enumerate: certify entire types using within-group largest-weight sums, then price their coverage against the smallest-weight sums. This gives a lower-bound certificate directly from the input weights. It need not count partially compatible types or solve an integer cover.

For a partition $\mathcal{P} = (I_g)_{g=1}^G$, retain the prefix extrema $s_g^-(j)$, $s_g^+(j)$ of Appendix C. Define

$$\mathcal{D}^- = \{b : \sum_g b_g = k, \sum_g s_g^+(b_g) \leq T\}, \quad \mathcal{T}^+ = \{c : \sum_g s_g^-(c_g) \leq T\},$$

with integer coordinates between zero and E_g , and set $N_b = \prod_g \binom{E_g}{b_g}$ and $K_{bc} = \prod_g \binom{c_g}{b_g}$. Every one of the N_b batches of a type $b \in \mathcal{D}^-$ is truly compatible. The sets of type $c \in \mathcal{T}^+$ include every actual cancellation set. The latter condition says some realization of c is feasible; it does not assert that all its realizations are feasible.

Proposition 4 (Input-only budget and omission certificate). *Let $\mathcal{S} \subseteq \mathcal{D}^-$ and let rational $y_b \geq 0$ satisfy*

$$\sum_{b \in \mathcal{S}} \frac{K_{bc}}{N_b} y_b \leq 1 \quad \text{for every } c \in \mathcal{T}^+. \quad (28)$$

Then $L = \sum_{b \in \mathcal{S}} y_b$ is a lower bound on the number of completions required to cover the entire true compatibility family. When $L > 0$, choose b with probability y_b/L , and independently choose b_g coordinates uniformly from each group I_g . All resulting batches are compatible, and any library of M legal completions omits the fixed pair on probability at least $\max\{0, 1 - M/L\}$.

For fixed G and rational input weights, an optimal such certificate can be chosen with polynomial encoding length. Any supplied rational certificate admits verification in polynomial bit complexity in its own length and the input length. More explicitly, if $S = |\mathcal{S}|$, enumeration of at most $\prod_g (E_g + 1)$ column types verifies it in $O(E \log E + SG \prod_g (E_g + 1))$ arithmetic operations, apart from standard binomial precomputation. The bit cost is polynomial in the input and certificate bit lengths. Neither computing actual type counts n_b nor enumerating compatible batches is required.

Proof. Every supported type is fully compatible by the largest-weight test. A completion of type c covers exactly K_{bc} of its N_b batches, so its mass under the stated distribution is at most $1/L$ by (28). The union bound proves both conclusions. Groupwise sorting computes all prefix extrema, and enumerating integer type vectors checks both support feasibility and every pricing constraint. Binomial coefficients have $O(E)$ bits. Exact sums and comparisons of rationals therefore have polynomial bit cost in the dimensions and encoded certificate length. There are at most $(E + 1)^G$ potential support rows and columns, giving polynomial certificate length for fixed G whenever its coefficients have polynomial bit length.

Such coefficients suffice even for an optimal supported dual: in the unnormalized variables $\lambda_b = y_b/N_b$, its constraint matrix is integer, with entries bounded by 2^E . A basic optimal solution, when the support is nonempty, uses rational coefficients whose numerators and denominators are determinants of matrices of dimension at most the number of supported rows. Hadamard’s inequality bounds their bit lengths by a polynomial in that row count and E . Multiplication by N_b preserves polynomial size. The LP is finite, feasible at zero, and bounded because each supported batch is coverable. $\square$

The certificate distribution can be sampled directly using its groups and supported types. It remains an explicitly constructed worst-case batch distribution, not an assertion about deployment noise or uniform sampling from the whole compatibility family. The omitted object is the fixed pair’s ambiguity evidence; other pairs may still provide risk bounds. The statement also holds for a randomized library drawn independently of the subsequent query batch.

One type is often sufficient. For any certified type b , define

$$q_b = \frac{N_b}{\max_{c \in \mathcal{T}^+} K_{bc}}, \quad q_{\mathcal{P}}^- = \max_{b \in \mathcal{D}^-} q_b. \quad (29)$$

The singleton-support choice $y_b = q_b$ is a certificate. Verifying one such certificate takes $O(G \prod_g (E_g + 1))$ arithmetic operations after prefix and binomial precomputation. A multi-type LP may improve the bound, but is unnecessary when a single certified type already resolves the specified budget. If $\mathcal{D}^-$ is empty, this test is inconclusive, even if many partially compatible types exist. These are lower bounds, with no general approximation factor or constructive upper guarantee.

What succinctness does and does not establish. This verification workload depends on group types rather than the number of compatible batches. It contrasts with a packing certificate that explicitly lists pairwise incompatible batches: a listed clique of size K contains K batch records. It does not prove that finding a large clique is hard on these instances, that symbolic packing certificates cannot also be compact, or that the type bound dominates packing. On the paper’s small enumerated continuous instances the tested clique baseline is stronger than the four-group LP, so that experiment cannot establish an advantage for that coarse LP. The input-only test addresses the regime in which the batch list is not materialized.

F CONSERVATIVE DEMAND COUNTING WITH UNCHANGED COMPLETION GEOMETRY

The purpose of this procedure is to retain some demands from a type whose minimum and maximum prefix sums straddle T . It does not approximate the set of allowed completions. All completion columns and all dual checks continue to use the original weights and threshold.

Fix a partition, $T > 0$, and an integer $Q > 0$. Define rounded costs $a_e^+ = \lceil Qw_e/T \rceil$ and $a_e^- = \lfloor Qw_e/T \rfloor$. For each size- k type b , let

$$\mathcal{F}_b^- = \{B : \text{type}(B) = b, \sum_{e \in B} a_e^- \leq Q\}, \quad \mathcal{F}_b^+ = \{B : \text{type}(B) = b, \sum_{e \in B} a_e^+ \leq Q\}.$$

Write $n_b^- = |\mathcal{F}_b^-|$ and $n_b^+ = |\mathcal{F}_b^+|$. If the exact maximum-prefix test certifies every batch of a type, replace both rounded families by that full type, so both counts are N_b . If the exact minimum-prefix test excludes it, replace both families by the empty set, so both counts are zero. These replacements only tighten the following bounds.

Proposition 5 (Certified partial demand counts). *For nonnegative rational weights, $\mathcal{F}_b^- \subseteq \mathcal{F}_b \subseteq \mathcal{F}_b^+$ and $n_b^- \leq n_b \leq n_b^+$. Let $D(r)$ denote the type LP with demand right-hand side r and the original completion columns. Then*

$$D(n^-) \leq D(n) \leq D(n^+), \quad D(n^-) \leq \tau^* \leq \chi_k.$$

The first interval bounds uncertainty in the counted type LP; $D(n^+)$ is not asserted to upper-bound τ^ or χ_k . Under $Q \mapsto 2Q$, the lower subfamilies grow and the upper subfamilies shrink. Every size- k batch with $\sum_{e \in B} w_e \leq T(1 - k/Q)$ is in the lower subfamily.*

Proof. The inequalities $Qw_e/T \leq a_e^+$ and $a_e^- \leq Qw_e/T$ give the two inclusions. Every original feasible completion of type c contains at most K_{bc} batches of either the full demand type or a retained subfamily. Consequently any $\lambda \geq 0$ satisfying $\sum_b K_{bc}\lambda_b \leq 1$ is a feasible cover dual when it assigns weight λ_b to retained batches of type b , and zero elsewhere. Its value is $\sum_b n_b^- \lambda_b$. Monotonicity of the objective in the nonnegative count vector gives the displayed inequalities.

For any real x , $\lceil 2x \rceil \leq 2\lceil x \rceil$ and $\lfloor 2x \rfloor \geq 2\lfloor x \rfloor$. A lower-feasible batch at resolution Q is therefore lower-feasible at $2Q$; upper feasibility at $2Q$ implies upper feasibility at Q . Finally, $\sum_{e \in B} a_e^+ \leq Q \sum_{e \in B} w_e/T + k \leq Q$ under the stated interior-margin condition. $\square$

Exact finite computation. Within group g , use a table $f_g(j, s)$ counting subsets with cardinality $j \leq k$ and rounded total $s \leq Q$. Insert each coordinate by the usual descending-cardinality update $f_g(j, s) \leftarrow f_g(j, s) + f_g(j-1, s-a_e)$. Coordinates with rounded cost above Q cannot belong to the corresponding rounded feasible family. Multiplying the group polynomials for the selected cardinalities b_g , truncating above degree Q , and summing coefficients gives the exact typed count. All coefficients are integers. The implementation uses signed 64-bit arithmetic only when $\max_{0 \leq j \leq k} \binom{E}{j}$ fits; otherwise it uses arbitrary-precision integers. This bound covers every intermediate convolution coefficient because groups contain disjoint coordinates. Zero costs use the same descending-cardinality update. For $T = 0$, only zero-weight coordinates can occur, so binomial counts give the answer directly.

With R retained demand types, direct polynomial convolution costs $O(EkQ + RGQ^2)$ arithmetic operations and $O(GkQ)$ working storage, apart from storing the output counts. This is a finite-resolution procedure, not a fully polynomial relative approximation guarantee. LP construction and exact certificate recovery have additional costs and explicit caps.

Threshold equality can remain unresolved. Take one group with $w = (1/3, 2/3, 2/3)$, $T = 1$, and $k = 2$. Its two compatible pairs both exhaust the threshold; the heavy pair is infeasible, so the type is partially compatible. For every $Q = 2^j$, each compatible pair has upward-rounded cost $Q + 1$. The lower count therefore remains zero although the true count and cover number are two. This establishes why rounding gives no unconditional convergence at threshold equality, even with the exact prefix bypass. We preserve both count bounds and do not present their ratio as a certified full-family cover approximation.

Controls and reproducibility. The rounded single-row control is $q_Q = \max_b n_b^- / \max_c K_{bc}$, and aggregate counting uses $\sum_b n_b^- / \binom{m}{k}$ with the exact maximum feasible cardinality m . Both use the same count computation as the joint LP. A gain over the prefix-only method is therefore not automatically a gain from combining rows. The exploratory study uses all 164 small-instance and 30 larger-instance inputs at the prespecified grid $Q \in \{64, 256, 1024\}$; the optimized-witness and size-sweep evaluations fix $Q = 256$. Exact demand enumeration on small instances is an independent validation check and is never an input to this method.

G JOINT INFORMATION AFTER MANDATORY-COLUMN PREPROCESSING

Lemma 1 (Private-demand columns and the residual problem). *Let $\mathcal{D}$ be a finite demand family and $\mathcal{C}$ a finite covering family, with every demand covered by some column. Write $\chi(\mathcal{D}, \mathcal{C})$ and $\tau(\mathcal{D}, \mathcal{C})$ for the integer and fractional minimum-cardinality cover values. Let $\omega(\mathcal{D}, \mathcal{C})$ be the maximum number of demands no two of which are jointly covered by one column.*

Call a demand private to $\mathcal{C}$ if $\mathcal{C}$ is its unique covering column. Let $\mathcal{S} \subseteq \mathcal{C}$ be all columns having a private demand, put $F = |\mathcal{S}|$, and define

$$\mathcal{R} = \{B \in \mathcal{D} : B \text{ is covered by some } C \in \mathcal{S}\}, \quad \mathcal{D}' = \mathcal{D} \setminus \mathcal{R}, \quad \mathcal{C}' = \mathcal{C} \setminus \mathcal{S}.$$

Then, with all three values defined as zero for an empty demand family,

$$\begin{aligned} \chi(\mathcal{D}, \mathcal{C}) &= F + \chi(\mathcal{D}', \mathcal{C}'), \\ \tau(\mathcal{D}, \mathcal{C}) &= F + \tau(\mathcal{D}', \mathcal{C}'), \\ \omega(\mathcal{D}, \mathcal{C}) &= F + \omega(\mathcal{D}', \mathcal{C}'). \end{aligned}$$

Every remaining demand has exactly the same covering columns as before the reduction. Applying this private-demand rule again therefore finds no additional forced column, unless another reduction changes the column family or its incidences.

Proof. For each $C \in \mathcal{S}$, fix a private demand B_C . Its cover constraint is $x_C \geq 1$. Thus every integer cover includes all F columns, and every fractional cover pays at least one unit for each. No column in $\mathcal{S}$ covers a remaining demand. Reducing any $x_C > 1$ to one preserves every cover constraint, since every affected row is already covered to level at least one by that column alone. After fixing these coefficients to one, the constraints for $\mathcal{R}$ are satisfied, and the remaining optimization problem is precisely the cover problem on $(\mathcal{D}', \mathcal{C}')$. Conversely, adding all forced columns to any residual cover covers the original family. This proves the integer and fractional identities.

The private demands B_C , one per distinct forced column, are pairwise incompatible: a column covering B_C must be C , and cannot also cover the private demand of another column. Each B_C is also incompatible with every residual demand, because C covers no residual demand. They therefore extend any residual packing by F demands, proving the lower inequality for ω . For the reverse inequality, any packing in the original problem contains at most F demands in $\mathcal{R}$: those rows are covered by the F forced columns, and each such column contains at most one member of a packing. The other members form a packing of the residual problem. This proves equality.

Finally, a remaining row had no incidence with any removed column; otherwise it would have been removed. Its degree and the identities of all columns incident to it are unchanged. Every degree-one row was already removed in the first pass, which proves the last assertion. $\square$

Application to knapsack completion. One may first restrict the columns to all inclusion-maximal feasible coordinate sets. Every feasible set extends to such a set, so column domination preserves χ , τ and ω . “Private” must be tested with respect to this maximal family: a demand can have a unique maximal completion without being an inclusion-maximal size- k demand. The lemma concerns unit-cost covers. It does not say that further residual column domination, column merging or other preprocessing can never expose additional reductions.

For a grouped residual counting comparator, count the actual remaining demands of each type, n'_b , and let

$$K'_{bC} = |\{B \in \mathcal{D}' : \text{type}(B) = b, B \text{ is covered by } C\}|.$$

The sound additive single-type bound is

$$\chi \geq \tau \geq F + \max_{b: n'_b > 0} \frac{n'_b}{\max_{C \in \mathcal{C}'} K'_{bC}},$$

with residual maximum zero for empty demand. Residual capacities count actual surviving incidences, since deleted rows can break type symmetry. No new LP or approximation principle is claimed by this preprocessing.

G.1 JOINT PRICES BEYOND PACKING AND EVERY COARSE-TYPE UNION

The next construction separates the information in joint prices from both optimal pairwise packing and a stronger counting control. This control may choose any union of the seven original demand types and assign every demand in that union the same price. It is not allowed to choose an arbitrary subset of individual demands; that larger comparison is not asserted. The construction has exact type symmetry, so its LP reduction uses the classical symmetrization argument.

Proposition 6 (Seven rows with a growing packing gap). *For every integer $n \geq 10$, take $4n$ light weights $1 + i/(10^6 n)$, $1 \leq i \leq 4n$, and n heavy weights $3/2 + j/(10^6 n)$, $1 \leq j \leq n$. Let $k = 6$, $T = 1051/100$. Every weight is distinct, all demand and completion types are invariant, and no maximal completion is mandatory. Let $D_n = U_n = \tau_n^*$ be the two-group fractional cover, q_n the strongest single-type bound, v_n the strongest uniform-price bound over any union of these seven types, and ω_n optimal pairwise packing. Then*

$$D_n = \frac{44893}{315} \frac{n^6}{720} + O(n^5), \quad q_n = \frac{256}{3} \frac{n^6}{720} + O(n^5), \quad v_n = \frac{183}{2} \frac{n^6}{720} + O(n^5), \quad \omega_n \leq \binom{5n}{5}/6.$$

Hence $D_n/\omega_n = \Omega(n)$ and $D_n/\max\{v_n, \omega_n\} \rightarrow 89786/57645 > 1.557$. Every single-type bound under a refinement of the two groups is at most q_n . Mandatory-column preprocessing changes nothing.

Proof. First ignore the small perturbations. All six-coordinate demands and all seven-coordinate sets are feasible. The maximal completion types, in light/heavy order, are

$$(10, 0), (9, 1), (7, 2), (6, 3), (4, 4), (3, 5), (1, 6), (0, 7).$$

Every feasible type costs at most 10.5, and the next possible infeasible cost is at least 11. A feasible set has at most ten coordinates, each perturbed by at most $4 \cdot 10^{-6}$. Thus every realization of a feasible type still costs less than 10.51; positive perturbations keep every infeasible type infeasible. This proves exact invariance with distinct weights.

Index demand rows by the number $b = 0, \dots, 6$ of light coordinates. Their counts are $N_b(n) = \binom{4n}{b} \binom{n}{6-b}$ and satisfy $N_b(n) = A_b n^6/720 + O(n^5)$, where $A = (1, 24, 240, 1280, 3840, 6144, 4096)$. In the displayed column order, the unnormalized incidence matrix $K_{b,(l,h)} = \binom{l}{b} \binom{h}{6-b}$ is

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 & 15 & 0 & 0 \\ 0 & 0 & 0 & 20 & 16 & 10 & 0 & 0 \\ 0 & 0 & 35 & 45 & 6 & 0 & 0 & 0 \\ 0 & 126 & 42 & 18 & 0 & 0 & 0 & 0 \\ 210 & 84 & 7 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The vectors

$$\lambda = (1/7, 0, 1/15, 0, 191/9450, 1/210, 1/210), \\ x = (1408/315, 256/7, 0, 256/3, 0, 16, 0, 1/7)$$

satisfy $K^\top \lambda \leq \mathbf{1}$, $Kx \geq A$, and $A^\top \lambda = \mathbf{1}^\top x = 44893/315$. For finite n , the same dual gives $N(n)^\top \lambda = (44893/315)n^6/720 + O(n^5)$, while $(n^6/720)x$ is feasible because $N_b(n) \leq A_b n^6/720$. This proves the leading value without assuming a numerical LP basis remains optimal at finite n .

Taking $\max_b A_b / \max_c K_{bc}$ gives $256/3$. For the stronger uniform-price control, a nonempty row subset S gives $\sum_{b \in S} N_b(n) / \max_c \sum_{b \in S} K_{bc}$. Substitution into the seven-by-eight matrix over its $2^7 - 1 = 127$ nonempty subsets gives leading maximum $183/2$, attained at $S = \{0, 1, 2, 3, 4, 5\}$. The accompanying exact checker retains every subset numerator and denominator. Since this is a finite maximum, the same leading expansion holds for v_n .

For a refined demand type inside coarse type b , choose a coarse completion type attaining $\max_c K_{bc}$ and distribute its realizations uniformly. Every concrete demand of type b is covered with probability $\max_c K_{bc} / N_b(n)$. Some refined completion type therefore covers at least that fraction of the refined row. Its single-row bound is no larger than $N_b(n) / \max_c K_{bc} \leq q_n$.

Two six-element batches sharing five coordinates have a seven-element union, which is feasible. In a pairwise incompatible collection, every five-element subset occurs in at most one batch. Each batch contains six such subsets, proving $\omega_n \leq \binom{5n}{5} / 6 = O(n^5)$. Finally, for any batch B and maximal completion $C \supseteq B$, choose a coordinate outside C ; one exists since $|C| \leq 10 < 5n$. The seven-set extending B by that coordinate is feasible and extends to another maximal completion. No batch is private to C . Combining the asymptotic bounds proves the claimed ratios. $\square$

For $n = 32, 64, 256, 1024$, the displayed feasible dual alone exceeds the universal packing upper bound by factors at least 1.409, 2.868, 11.624 and 46.649. Its ratios to the best exact coarse-type-union count are at least 1.539, 1.548, 1.555 and 1.557. These are algebraically checked constructed examples, not empirical cohorts. They do not separate from arbitrary implicit covering methods or uniform prices on arbitrary individual-demand subsets.

G.2 A TWO-GROUP REGIME WITHOUT MANDATORY COMPLETIONS

Proposition 7 (Two-group separation without private demands). *For every even $n \geq 6$, set $T = 41/10$, $k = 2$, and use the distinct light and heavy weights*

$$\ell_0 = 1, \quad \ell_1 = 1 + \frac{1}{1000n}, \quad \ell_j = \frac{26}{25} + \frac{j-2}{200n} \quad (2 \leq j < n), \quad h_j = \frac{3}{2} + \frac{j}{100n} \quad (0 \leq j < n).$$

Partition by light and heavy coordinates, and write $H = \binom{n}{2}$. Every pair demand is feasible and has at least two distinct maximal feasible completions. The optimistic and robust type-cover values are

$$D_{\mathcal{P}} = \frac{7n^2 - 2n}{12}, \quad U_{\mathcal{P}} = \frac{2n^2 - n}{3}, \quad D_{\mathcal{P}} \leq \tau^* \leq U_{\mathcal{P}}.$$

The optimal pairwise packing, strongest single-type counting, and aggregate counting values are

$$\omega_{\text{pair}} = \frac{n^2}{2} - 1, \quad q_{\mathcal{P}} = \frac{n^2}{2}, \quad a = \frac{n(2n-1)}{6}.$$

Thus private-demand forced-column preprocessing removes nothing, and

$$\frac{D_{\mathcal{P}}}{\max\{\omega_{\text{pair}}, q_{\mathcal{P}}, a\}} = \frac{7}{6} - \frac{1}{3n} \rightarrow \frac{7}{6}, \quad \frac{U_{\mathcal{P}}}{D_{\mathcal{P}}} = \frac{4(2n-1)}{7n-2} \rightarrow \frac{8}{7}.$$

Moreover the actual full fractional optimum satisfies

$$\max \left\{ D_{\mathcal{P}}, U_{\mathcal{P}} - \frac{2n-3}{3} \right\} \leq \tau^* \leq U_{\mathcal{P}}, \quad \frac{\tau^*}{D_{\mathcal{P}}} \rightarrow \frac{8}{7}.$$

These are fractional-cover brackets; equality with the integer cover number is not asserted.

Proof. All lights weigh at least one and less than 1.045, while all heavies weigh at least 1.5 and less than 1.51. Thus every pair is feasible, every LLH set weighs less than 3.60, and every LHH set weighs less than 4.065. In contrast, $LLLH$, HHH , $LLHH$ and five-light sets are infeasible by their minimum weights. Four lights are feasible exactly when they include both cheap lights ℓ_0, ℓ_1 : without both their weight is at least 4.12, while the greatest weight with both is $4.09 - 17/(500n) < T$. All three-light sets are feasible. Consequently the maximal minimum-prefix types are $(4, 0), (2, 1), (1, 2)$, and the maximal maximum-prefix types are $(3, 0), (2, 1), (1, 2)$. Demand types are stable and completion types are ambiguous.

In row order LL, LH, HH and the displayed column orders, the two coverage-count matrices and common demand counts are

$$K^- = \begin{pmatrix} 6 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad K^+ = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad N = (H, n^2, H)^\top.$$

For the optimistic LP, the nonnegative column masses

$$x^- = ((H - n/2)/6, n/2, H)$$

satisfy every row with equality. The per-demand dual coefficients $\lambda^- = (1/6, 5/12, 1/6)$ give unit value on every column. Their matching objective is $(7n^2 - 2n)/12$. For the robust LP use

$$x^+ = ((H - n/2)/3, n/2, H), \quad \lambda^+ = (1/3, 1/3, 1/3).$$

Again all row constraints and column dual constraints are tight, with objective $(2n^2 - n)/3$. These exact primal–dual certificates establish the two values. Uniformly distributing the robust masses over all coordinate realizations gives a genuine fractional cover, establishing the stated full-family bracket.

No demand is private to a maximal completion. An HH pair extends to a distinct maximal LHH completion for every light. An LL pair extends to a distinct maximal LLH completion for every heavy. An LH pair has $n - 1$ distinct maximal LLH extensions (and also $n - 1$ distinct maximal LHH extensions). Hence the forced-column set is empty.

For packing, selected mixed demands form a matching between lights and heavies: two sharing a light have a feasible LHH union, and two sharing a heavy have a feasible LLH union. If the packing contains $s \geq 1$ mixed demands, it excludes all HH pairs incident to their s heavy coordinates and all LL pairs incident to their s light coordinates. The packed LL pairs are disjoint because three lights always fit. Its size is therefore at most

$$\binom{n-s}{2} + s + \left\lfloor \frac{n-s}{2} \right\rfloor \leq \frac{(n-s)^2}{2} + s.$$

The right-hand side is convex in $s \in [1, n]$, and both endpoint values $(n-1)^2/2 + 1$ and n are strictly less than $n^2/2 - 1$ for $n \geq 6$.

If there is no mixed demand, all H heavy pairs can be selected; they conflict with each other and every light pair. A light-pair packing is a matching, but cannot contain a perfect matching on all n lights. If the cheap lights share an edge, that edge has feasible union with any other matching edge. If they belong to different edges, those two edges have feasible union. Thus there are at most $(n-2)/2$ light pairs. A matching on the $n-2$ expensive lights attains that number, since any two such edges have union weight at least $4.16 > T$. Together with all heavy pairs this proves $\omega_{\text{pair}} = H + (n-2)/2 = n^2/2 - 1$.

The single-type ratios are $H/6, n^2/2, H$, giving the stated maximum. All $\binom{2n}{2} = n(2n-1)$ pairs are demands, and the greatest feasible completion size is four, proving the aggregate bound. For $n \geq 6$, q_P exceeds both other controls. Substitution gives the two displayed ratios.

For the sharper full-family lower bound, give per-demand dual mass $1/3$ to every HH and LH pair, and to every LL pair entirely among the expensive lights indexed by $j \geq 2$; give mass zero to other light pairs. A feasible pure-light triple has at most three such positive pairs. A feasible four-light set contains both cheap lights, so has only one positive pair. An LLH completion has at most one positive light pair and two mixed pairs; an LHH completion has two mixed pairs and one heavy pair. Every completion therefore has dual mass at most one. The objective is

$$\frac{\binom{n-2}{2} + n^2 + H}{3} = U_P - \frac{2n-3}{3}.$$

Together with the robust upper, this proves the claimed squeeze and the limiting actual aggregation loss. $\square$

Weighted conversion and actual loss. The optimistic primal uses mass $a = (H - n/2)/6$ on its four-light column, mass $n/2$ on LLH , and mass H on LHH . Replacing the first mass by $2a$ on

the robust three-light column preserves its light-pair coverage and leaves the other rows unchanged. This recovers exactly the robust objective $U_{\mathcal{P}}$. The worst-column retention is $1/2$, giving a generic factor of two, but this column-specific conversion has factor $U_{\mathcal{P}}/D_{\mathcal{P}} \rightarrow 8/7$. The full-family dual above shows that this smaller factor describes actual asymptotic aggregation loss, not just slack in the robust upper construction.

Scope. The single-type comparison also survives every refinement of the light/heavy partition. A refined HH row has capacity one and at most H demands (a cross-group row has at most $n^2/4$). A within-group LL row with group size at least three has capacity at least three; with size two its ratio is one. A cross-group LL row has capacity at least two unless both groups are singletons, when its ratio is one; its demand count is at most $n^2/4$. A refined LH row on groups of sizes a, b has capacity at least two unless $a = b = 1$, since both LLH and LHH types are robust. Its ratio is therefore at most $ab/2 \leq n^2/2$, or one in the singleton case. Thus no such refined row exceeds the original $q_{\mathcal{P}} = n^2/2$. Adding the one-group aggregate bound changes nothing. This covers nested group sequences whose two-group level is exactly the light/heavy partition, including their singleton endpoint; arbitrary cross-cutting partitions are not asserted here.

This is a constructed separation from the listed controls, including private-demand preprocessing followed by the same residual single-type bound. It is not a separation from every stronger cover reduction or implicit method, and it does not establish runtime dominance or prevalence in measured inputs. The LP duality and preprocessing rules are standard; the claim here is the explicit knapsack construction and its verified formulas.

H EQUAL BEST-COMPLETION SIZE CAN HIDE AN EXPONENTIAL COVER GAP

Maximum cancellation cardinality describes the best single completion's raw coverage, but does not characterize how its coverage overlaps the rest of the compatible family. The following two rank-one families make this distinction quantitative while keeping their compatible-batch counts within a factor of two.

Theorem 7 (A matched-cardinality separation). *For every positive even integer n , take $E = 4n$, $k = n$, $T = 2n$ and fixed prediction difference $z = 1$. Compare the homogeneous shape $\Sigma_{\text{hom}} = I_{4n}$ with $\Sigma_{\text{het}} = \text{diag}(I_{2n}, \frac{1}{3}I_{2n})$. Both have maximum feasible cancellation cardinality $m_* = 2n$, and hence the same maximum raw single-completion coverage $\binom{2n}{n}$. Let $N = \binom{4n}{n}$ and $D = \binom{2n}{n/2}^2$. Their compatible-batch counts are N and $(N + D)/2$, respectively, so they differ by at most a factor of two. Nevertheless their fixed-pair completion cover numbers obey*

$$\frac{N}{\binom{2n}{n}} \leq \chi_{\text{hom}} \leq \min \left\{ N, \left\lceil (1 + \log N) \frac{N}{\binom{2n}{n}} \right\rceil \right\}, \quad D \leq \chi_{\text{het}} \leq \frac{N + D}{2},$$

and, along even $n \rightarrow \infty$,

$$\lim_{4n} \frac{1}{4n} \log \frac{\chi_{\text{het}}}{\chi_{\text{hom}}} = \frac{1}{2} \log 2. \quad (30)$$

Both differences arise from $P = [0, 1]$, $q(\theta) = g_e(\theta) = \theta$ and the fixed pair $1, 0$, so prediction rank and target gap are identical. Their normalized full-menu radii are $\sqrt{2}$ and 2 , respectively.

Proof. Homogeneous cancellation costs one per coordinate. In the heterogeneous instance the first $2n$ coordinates cost one and the remaining $2n$ cost three. A cancellation set of size $2n$ is feasible by taking all light coordinates; every larger set costs at least $2n + 3 > T$. Thus both maxima equal $2n$.

A batch with j heavy coordinates costs $n + 2j$, so it is compatible exactly when $j \leq n/2$. Its type has $\binom{2n}{j} \binom{2n}{n-j}$ members. These counts are symmetric in j and $n - j$; summing over all types gives N by partitioning all n -subsets. The compatible half, including the central type of size D , therefore has $(N + D)/2$ members.

Every central-type batch uses the whole energy budget T . Since all coordinate costs are positive, an admissible completion covering it cannot cancel any additional coordinate. That completion covers only that batch among size- n batches. These D batches each require a distinct completion, giving

the heterogeneous lower bound. One completion per compatible batch gives its upper bound. The homogeneous bounds follow from the one-group fractional value and rounding theorem.

The binomial entropy estimates give

$$\log D = 4nH(1/4) + O(\log n), \quad \log N = 4nH(1/4) + O(\log n),$$

$$\log \binom{2n}{n} = 2n \log 2 + O(\log n).$$

The displayed upper and lower bounds thus determine both cover exponents; subtracting them proves (30). Finally $Q_z([E])/T$ is 2 or 4. The largest absolute model difference in $P - P$ is one, giving the two normalized radii. $\square$

The maximum-coverage cardinality control and the aggregate counting lower bound $|\mathcal{F}_{z,k}|/\binom{m_*}{k}$ cannot distinguish these cover rates: the latter changes by at most a factor of two. The mixed batch type in the orbit profile does distinguish them. This is a fixed-pair representation result. It does not bound a search free to change model pairs, nor assert a runtime advantage. The heterogeneous hard type is energy-saturated; the theorem does not claim robustness to arbitrary fixed relative changes of T .

I A MATCHED-COVERAGE GAP WITH ALL WEIGHTS DISTINCT

The exact matching example uses two weight values, but its exponential separation is not confined to an equality subspace. The following family preserves identical best single-completion coverage for arbitrary distinct weights within fixed-width intervals. The factor-two matching of total compatible-family sizes from Theorem 7 is not asserted here.

Theorem 8 (Distinct weights and equal maximum coverage). *Fix $\eta = 1/100$. For each positive integer n , choose any distinct weights $h_1 < \dots < h_{4n}$ in $[1, 1 + \eta]$, and set $T_n = \sum_{i=1}^{2n} h_i$. The homogeneous-neighborhood instance has weights $(h_i)_{i=1}^{4n}$. The heterogeneous-neighborhood instance has light weights $h_1, \dots, h_{2n}$ and any distinct heavy weights in $[3, 3 + \eta]$. Take $E = 4n$ and $k = n$ in both. Both instances have all weights distinct, maximum cancellation cardinality exactly $2n$, and maximum single-completion coverage $\binom{2n}{n}$. Nevertheless, uniformly over these choices,*

$$\liminf_{n \rightarrow \infty} \frac{1}{4n} \log \frac{\chi_{\text{het}}}{\chi_{\text{hom}}} \geq \gamma_{\text{distinct}} > 0.3150, \quad (31)$$

where, writing $H(p) = -p \log p - (1 - p) \log(1 - p)$ and $\beta = 1/[2(1 + \eta)]$,

$$\begin{aligned} \gamma_{\text{distinct}} &= \frac{1}{2} H((1 - \eta)/4) + \frac{1}{2} H((1 + \eta)/4) - H(1/4) \\ &\quad + \beta H(1/(4\beta)) - \frac{1+\eta}{2} H(\eta/(1 + \eta)) = 0.31504998 \dots \end{aligned}$$

All weights arise from rank-one predictions $z = \mathbf{1}$ and diagonal shapes $\Sigma_{ee} = 1/w_e$.

Proof. In both instances the cheapest $2n$ coordinates have total weight T_n , and adding the next cheapest coordinate exceeds the budget. Thus the best cancellation cardinality is exactly $2n$ in each. Every n -batch in the homogeneous-neighborhood instance is compatible, since its energy is at most $(1 + \eta)n < 2n \leq T_n$. Every set of size $m_n = \lfloor T_n/(1 + \eta) \rfloor$ is feasible there. The classical uniform covering bound therefore gives

$$\chi_{\text{hom}} \leq \left\lceil \left(1 + \log \binom{4n}{n}\right) \frac{\binom{4n}{n}}{\binom{m_n}{n}} \right\rceil.$$

Since $m_n \geq \lfloor 2n/(1 + \eta) \rfloor$, the limsup of its log divided by $4n$ is at most $H(1/4) - \beta H(1/(4\beta))$.

For the heterogeneous instance, let $j_n = \lfloor [T_n - (1 + \eta)n]/2 \rfloor$. The batch type with j_n heavy and $n - j_n$ light coordinates is entirely compatible: its energy is at most $(1 + \eta)n + 2j_n \leq T_n$. Its size is $D_n = \binom{2n}{j_n} \binom{2n}{n-j_n}$, and

$$(1 - \eta)/2 - 1/n \leq j_n/n \leq (1 + \eta)/2.$$

Every batch of this type has energy at least $n + 2j_n > T_n - \eta n - 2$. Every additional canceled coordinate costs at least one, so any legal completion covering a batch of this type has fewer than $\eta n + 2$ extra coordinates. With $r_n = \lceil \eta n + 2 \rceil$, such a completion covers at most

$$\binom{n - j_n + r_n}{r_n} \binom{j_n + r_n}{r_n} \leq \binom{n + r_n}{r_n}^2$$

members of the hard type. Consequently $\chi_{\text{het}} \geq D_n / \binom{n+r_n}{r_n}^2$. The binomial entropy bounds give

$$\liminf \frac{\log D_n}{4n} \geq \frac{1}{2}H((1-\eta)/4) + \frac{1}{2}H((1+\eta)/4), \quad \lim \frac{\log \binom{n+r_n}{r_n}^2}{4n} = \frac{1+\eta}{2}H(\eta/(1+\eta)).$$

For the first bound, the symmetric concave function $[H(a/2) + H((1-a)/2)]/2$ attains its minimum over the limiting interval at an endpoint. Subtracting the homogeneous upper exponent proves the claim. The final construction takes a scalar model difference one and $\Sigma_{ee} = 1/w_e$. $\square$

This theorem controls independent perturbations of individual coordinates within fixed intervals, not just perturbation of a shared energy budget. The common budget is set to the energy of the $2n$ cheapest coordinates to keep maximum coverage matched. It is a representation result, and does not assert an optimization lower bound or a runtime separation.

J TRANSFER TO REUSABLE CONCRETE-MODEL CACHES

Concrete-model reuse is an operational interpretation of the covering incidence and duality. KLEE's counterexample cache uses stored assignments, while its wider solver chain also supports richer reasoning and fallback solving. The following result permits the pre-query model collection to change after every query.

Fix nonnegative weights $w \in \mathbb{R}_+^E$, a budget $T \geq 0$, and a nonempty family

$$\mathcal{F} = \{B \subseteq [E] : |B| = k, \sum_{e \in B} w_e \leq T\}.$$

For Boolean variables $x \in \{0, 1\}^E$, define the base constraint and positive-assumption queries

$$F_{w,T}(x) : \sum_{e=1}^E w_e x_e \leq T, \quad Q_B(x) : F_{w,T}(x) \wedge \bigwedge_{e \in B} (x_e = 1).$$

When an actual finite SMT encoding is intended, the weights and budget may be restricted to rationals and scaled to integers; the mathematical claims do not need that restriction. A cache entry is a complete concrete model of $F_{w,T}$. A cache hit occurs precisely when at least one entry satisfies Q_B . A partial, symbolic, or parametrized description standing for multiple assignments is outside this entry-counting convention.

Proposition 8 (Concrete-model cache equivalence). *The minimum number of concrete models whose cache answers every satisfiable size- k positive-assumption query is $\chi_k(w, T)$. Consequently, a cache of at most M models can have complete coverage if and only if $M \geq \chi_k(w, T)$.*

Proof. Nonnegativity implies that Q_B is satisfiable if and only if $B \in \mathcal{F}$: its indicator vector supplies a model in one direction, and any satisfying assignment has weight at least $\sum_{e \in B} w_e$ in the other. The support $C = \{e : x_e = 1\}$ of a legal entry satisfies $\sum_{e \in C} w_e \leq T$. This entry answers Q_B exactly when $B \subseteq C$. Conversely, each feasible cancellation set C gives the entry $x = \mathbf{1}_C$. The two minimum-cardinality problems therefore have identical incidences. $\square$

Theorem 9 (Miss guarantee with adaptive cache replacement). *Let $\lambda_B \geq 0$, $B \in \mathcal{F}$, be any feasible full-cover dual:*

$$\sum_{B \in \mathcal{F} : B \subseteq C} \lambda_B \leq 1 \quad \text{for every } C \subseteq [E] \text{ with } \sum_{e \in C} w_e \leq T.$$

Write $D = \sum_B \lambda_B > 0$ and $\mu(B) = \lambda_B/D$. Consider any randomized cache procedure which, before query t , has at most M concrete models of $F_{w,T}$. These models can depend arbitrarily on

previous queries, answers, insertions, and evictions. Suppose that each B_t is sampled independently from μ , independently of the procedure’s randomness. If H_t indicates a concrete-model cache hit, then

$$\mathbb{E}[H_t \mid \mathcal{H}_{t-1}] \leq \min\{1, M/D\}, \quad \mathbb{E}\left[\sum_{t=1}^r (1 - H_t)\right] \geq r \max\{0, 1 - M/D\}.$$

Here $\mathcal{H}_{t-1}$ includes the selected pre-query cache as well as the preceding history. Thus the conclusion does not require a fixed cache or a particular replacement rule.

Proof. Condition on $\mathcal{H}_{t-1}$. Each available model has a feasible support C and, by dual feasibility, answers a set of queries of μ -mass at most $1/D$. A union bound over at most M entries bounds the hit probability by $\min\{1, M/D\}$. Independence makes the conditional law of B_t equal to μ , even though the entries depend on earlier queries. Taking expectations and summing proves the result. A newly constructed model after observing B_t can serve later queries, but does not turn the initial lookup for B_t into a cache hit. $\square$

For the partition dual in the main text, set $\lambda_B = y_b/n_b$ when B has type b . Its dual inequalities give exactly the feasibility condition above. For the input-only version, use $\lambda_B = y_b/N_b$ on fully certified types and zero elsewhere. Thus its difficult query distribution can be sampled directly by first sampling a type and then uniform within-group subsets. This construction does not assert that application queries follow that distribution.

Corollary 3 (Matched aggregate statistics, different cache capacity). *For every positive even n , take the homogeneous and heterogeneous instances of Theorem 7. Put*

$$N_n = \binom{4n}{n}, \quad S_n = \binom{2n}{n}, \quad L_n = \binom{2n}{n/2}^2, \quad M_n = \min\left\{N_n, \left\lceil (1 + \log N_n) \frac{N_n}{S_n} \right\rceil\right\}.$$

The homogeneous query family admits a preloaded concrete-model cache of at most M_n entries which answers every satisfiable query. For the heterogeneous family, let μ_n be uniform on the L_n queries selecting $n/2$ coordinates from each weight class. Every cache procedure with at most M_n pre-query models, including adaptive replacement, satisfies

$$\mathbb{E}\left[\sum_{t=1}^r H_t\right] \leq r \min\{1, M_n/L_n\}, \quad \frac{M_n}{L_n} = \exp\{-2n \log 2 + O(\log n)\}.$$

The two families have the same maximum single-model coverage S_n and total satisfiable-query counts within a factor of two.

Proof. The homogeneous covering upper bound supplies its cache by Proposition 8. Each heterogeneous query in the stated type exhausts the budget $T = 2n$. Because all weights are positive, any model answering it has exactly that query’s support and answers no other query in the type. Assigning dual weight one to these L_n queries is feasible. Apply Theorem 9 with $D = L_n$. The binomial estimates in Theorem 7 give the displayed rate and the matched statistics. $\square$

Meaning and limits. A miss requires an answer outside the current model pool, not necessarily an SMT invocation: simplification or symbolic reasoning may suffice. Indeed, $x = \mathbf{1}_B$ answers this specialization in $O(k)$ sparse work. The theorem bounds neither KLEE’s total memory, solver calls nor runtime; KLEE need not impose capacity M . The matched construction concentrates on saturated queries and makes no empirical workload claim.

K SMALL-INSTANCE INPUTS, CONTROLS AND COMPLETE DECISION CENSUS

Inputs and dependence. The small-instance set contains 140 witnesses from all 48 diagonal sensing tasks and 24 Wine conditions. Continuous tasks use Gaussian or exponential-decay sensing, $P = [-1, 1]^p$, $c = .65$, $E = 8, 12$ and $k = 2, 3$. The 48 tasks share 12 seed-shape streams. Distinct first, middle and last certificate indices are selected; weights z_e^2/Σ_{ee} and $4c^2$ are serialized binary64 values interpreted as exact dyadic rationals. Thus exact cover certificates concern these serialized

Input source	M	$G = 4$	$G = 8$	Full	Strong+	$G = 8$ only	Full only
Continuous (140)	4	82	97	99	99	0	0
Continuous (140)	16	7	19	28	21	2	7
Wine (24)	4	9	12	15	11	1	4

Table 4: All prescribed-budget rejection counts. Blue highlights paired decisions absent from Strong+. The capped clique remains in totals and supports no claim against optimal packing. At $M = 1$, all methods reject 143/164 conditions; no Wine condition rejects $M = 16$.

inputs, not interval-certified real-valued oracle solutions. Of the 140 witnesses, 138 have all weights distinct. Source task generators and selected inputs are included in the reproduction package.

Wine (Aeberhard & Forina, 1992) uses the 178-row, 13-feature scikit-learn array. Features are min/max scaled over the full dictionary and quantized to integers in $[0, 16]$; class 0 versus the others defines the target. Balanced nested dictionaries of sizes 24 and 96 use seed 20260924 and $T \in \{16, 64\}$. First/middle/last cross-target pairs in each of four search streams, each at $k = 2, 3$, give 24 dependent conditions. This is a finite-dictionary geometry experiment, not a prediction or noise calibration experiment. Conditions and repeated analyses are not treated as independent samples.

Certificates and comparisons. Fixed partitions sort by weight/index, then repeatedly bisect a largest group, breaking ties from the left. We evaluate $G = 1, 2, 4, 8, E$. Each method regenerates its family, groups, incidences or graph from (w, T, k) within its clock. Componentwise dominated completion types may be omitted because all nonnegative dual loads increase with their entries. SciPy/HiGHS proposes LP solutions; nonnegative duals are rationalized with denominator limit 10^6 and scaled by their maximum exact column load if it exceeds one. Primals are independently scaled to cover every row using exact arithmetic. Singleton refinement is an enumerated reference. Three rotated repetitions use one worker and one numerical thread, with two-second internal LP and maximum-clique caps. All LPs finish; the same clique closes in 163/164 cases. The capped case remains in all totals.

Strong+ takes the maximum of aggregate counting, pairwise packing, single-type bounds at $G = 1, 2, 4, 8$, and mandatory-column preprocessing. The latter enumerates maximal completions, charges every column with a private request, removes their covered requests and recomputes residual counting from actual incidences. Mandatory columns occur in 155 cases and strengthen ten bounds, but change no prescribed-budget decision. The control amendments were specified before their respective executions and remain exploratory analyses of the same small-instance set.

G4 adds no prescribed-budget rejection, G8 adds three and full refinement adds eleven. Full refinement exceeds optimal packing on 53 of the 163 closed conditions; G8 does so on 13 and G4 on five. The 53 gaps span 51 geometries and 32 tasks, with minimum gap $1/7$. The eleven full-only budget decisions span ten geometries and eight tasks. These are different events from the greedy exact-count closures in Appendix L.1. Median end-to-end times are 1.842 ms (G4), 3.045 ms (G8), 6.380 ms (full), and 3.494 ms for the actual joint Strong+ implementation with shared input work. Standalone optimal/capped packing costs .658 ms. Imports, input/output and upstream witness generation are excluded. Separate standalone medians are not summed into a portfolio speed ratio.

The three G8-only decisions. A and B are exponential-decay diagonal tasks with $E = 12, p = 3, k = 2$: seed 809, certificate 39 (A), and seed 1103, certificate 12 (B). C is Wine, dictionary size 96, $T = 64$, certificate 13, $k = 3$. Their exact G8 bounds are $33/2, 52/3, 34/7$ against Strong+ values 16, 16, 4. Their checked greedy integer uppers are 20, 22, 7; none closes an exact integer count. A and C have fractional brackets with ratios 1.061 and 1.178; B has unstable demand and no such upper. Mandatory preprocessing charges 3, 9, 0 columns and leaves 60, 22, 275 requests, without removing these three additional decisions.

K.1 DIRECT BRACKETS AND ANALYTIC CONTRACTION

Fixed G4 and G8 have stable demand in 19 and 39 conditions. Removing exact type-invariant cases leaves seven and 16 with ambiguous completion types. Their direct robust upper/lower ratios have medians 2.066 and 1.301, maxima 4 and 2.365, and fall within two in $2/7$ and $15/16$ cases. On the 16

E	Compatible batches	D	U	Strong+	$D/\text{Strong+}$	Time (ms)
12	66	20	22	18	1.111	2.96
32	496	146.67	165.33	128	1.146	2.87
128	8,128	2,378.67	2,709.33	2,048	1.161	2.92
512	130,816	38,186.67	43,605.33	32,768	1.165	3.04
2,048	2,096,128	611,498.67	698,709.33	524,288	1.166	3.45

Table 5: Constructed scaling with no mandatory column. Blue marks the joint lower’s improvement over the combined counting and optimal-packing control. U is fractional; the experiment does not compute integer optima or estimate prevalence. Values are rounded only for display.

nontrivial G8 cases, uniform and column-specific contraction of the same checked lower-LP *primal* give median ratios 8.75 and 4.077. These are feasible-primal conversions, not weighted-LP optima; dividing a numerical dual lower bound by ρ would not certify an upper bound. The direct robust LP is materially tighter here. No unstable case has a full-family upper. Independent enumeration checks 164 inputs, 328 upper libraries and 506,780 actual completion–dual inequalities.

K.2 CONSTRUCTED SCALE TEST

The forced-free family uses $E = 12, 32, 128, 512, 2048$, fixed before measurement. Both LPs have three rows and three maximal columns. Each of three runs charges exact weight construction, sorting, pruned type enumeration, both solves and exact rational checks. No batches or conflict edges are materialized. Packing uses its proved value, not a timed graph solver.

L CANONICAL GROUPING CENSUS AND BOUNDARY STRESS TEST

Scope and grouping rule. The census includes all 164 small-instance conditions. Fixed partitioning uses equal-sized weight groups; canonical grouping follows Section 3.4, with deterministic largest-block median splits and leftmost ties. Outcomes are not consulted when splitting. All refusals remain in the denominator.

The run evaluates fixed groups, the unfilled minimum partition, and the canonical partition filled to four or eight groups. Three repetitions rotate method order on one worker and one numerical thread. The clock includes the minimum-partition calculation when used, type construction, both LP solves, rational recovery and exact certificate checks; imports and input/output are excluded. Each LP has a two-second limit. Every applicable LP completes, and rational bounds are invariant across repetitions. Exact lower primal–dual equality is checked for every applicable canonical result. The census and its fill-rule amendment were specified before their respective executions, but reuse the small-instance set and are not an independent confirmation cohort.

Applicability and precision are separate outcomes. Table 1 gives all applicability and precision counts. The seven applicable eight-group brackets above ratio two reach 3.5; these differ from the seven cases where the cap is insufficient. Neither an exact fractional bracket nor a cap refusal constructs an integer library.

The main interval panel displays every canonical eight-group bracket after division by the same condition’s checked full singleton-LP lower reference. This normalizes scales without asserting that every stored reference is an exact optimum: three full-LP rational primal recoveries have small gaps. The reference lies inside every displayed interval. Vertical intervals are deterministic certificates, not statistical confidence intervals. Valid conditions are sorted by U/D ; the seven unavailable cases are shown as a separate gray state after them. Black bottom ticks identify the 39 cases where fixed eight-group bracketing is applicable.

The canonical rule does not dominate every fixed partition. Table 6 compares only commonly applicable conditions. The near-one paired timing ratios do not support a speedup claim from the methods’ different successful subsets.

G	Common	Better/equal/worse	Fixed ms	Canonical ms	Paired ratio	Added valid
4	19	1/18/0	2.55	2.62	1.012	59
8	39	3/32/4	3.83	3.90	1.011	118

Table 6: Paired comparison on conditions applicable to both grouping rules.

“Better/equal/worse” compares U/D for canonical and fixed grouping; smaller is better. Each time first takes the median of three repetitions within a condition, then the median over the common subset. The paired ratio is the median within-condition canonical/fixed time ratio.

Filling the available group budget also matters. The unfilled minimum partition is applicable on the same 157 cases, but only 73 have $U/D \leq 1.25$, 111 have $U/D \leq 2$, and 60 have $D = U$. Its median ratio is approximately 1.455. Refinement improves these counts to 107, 150, and 79 without changing the applicability set. This ablation supports using available group capacity after computing the minimum; it does not guarantee a tight bracket at every cap.

Larger input regimes can require substantially more groups. An exact post-hoc diagnostic evaluates all 16 random-direction stress-test conditions, comprising eight Gaussian linear-pair and eight Digits image-pair cases with $E \in \{24, 48\}$ and $k \in \{4, 6\}$. Their minimum group counts range from 7 to 48; only two permit a demand-stable partition with eight groups, and none permits four. In contrast, the small-instance $E \in \{8, 12, 13\}$ conditions have minimum counts from 1 to 11. The figure shows exact empirical cumulative distributions for these cohorts and the 48 optimized-witness evaluation inputs, all of which require more than eight groups. These are different input regimes, so the curves do not estimate a common population prevalence or supply a paired performance comparison. The diagnostic was computed after the random-direction stress test; it motivates the partial-counting fallback and is excluded from that test’s prespecified method comparison.

L.1 COMBINING LOWER BOUNDS WITH AN INTEGER GREEDY LIBRARY

The lower-bound comparisons ask which lower certificate rejects a prescribed capacity. To measure integer sizing, we implement the classical remaining-coverage greedy construction: at each step select a completion covering the largest number of requests not yet covered (Kurtz, 2026, Section 5). On the 164 small-instance inputs, $E \leq 13$ allows enumeration of every feasible maximal completion, so the remaining-coverage oracle is exact. Ties choose the smallest integer support mask. There are no restarts or selection among multiple greedy libraries. This is a baseline construction, not a new algorithm.

Let K_{gr} be the resulting checked library size and let L be any valid lower bound. Then

$$\lceil L \rceil \leq \chi_k \leq K_{\text{gr}}.$$

Equality determines the exact integer optimum. Otherwise $K_{\text{gr}} - \lceil L \rceil$ counts the integer capacities left unresolved: those below $\lceil L \rceil$ are rejected and those at least K_{gr} permit export. Every portfolio uses the same checked upper; the LP portfolios maximize Strong+ and the named LP lower. Consequently a stronger lower cannot add feasible/export decisions in this comparison.

Table 2 summarizes the paired integer intervals.

Strong+ and the integer upper determine 103 exact policy counts. G8 narrows ten intervals and removes twelve unresolved integer capacities, but determines no additional exact counts. G4 narrows two intervals. Full refinement determines eleven additional exact counts and removes 66 unresolved capacities. Its eleven extra exact counts span ten distinct geometries and nine source tasks; two Wine conditions duplicate one geometry. These exact-count closures are different events from the eleven full-only prescribed-budget rejections. Full refinement remains an enumerated reference.

At $M = 4$, the feasible/rejected/unresolved totals are 48/110/6 for Strong+, 48/111/5 after G8, and 48/114/2 after full refinement. At $M = 16$ they are 133/21/10, 133/23/8, and 133/28/3, respectively. G4 does not change these decisions. At $M = 1$, every portfolio yields 21/143/0. Thus G8 adds the same three prescribed-budget rejections; the integer upper does not create additional favorable G8 decisions.

Capped packing and a greedy-derived dual. The one condition whose maximum-clique search remained capped is included in all totals. On the 163 closed-clique conditions the interval-width sums are 175, 173, 163, 114 for Strong+, G4, G8 and full refinement. The capped condition changes from $[26, 33]$ under Strong+ to $[31, 33]$ with the full LP. This is a gain over its verified packing lower, not over an unknown optimal clique.

As a secondary classical control, give every newly covered request price $1/a$ when its greedy step covers a new requests. Divide all prices by their maximum load over every maximal completion. The resulting nonnegative request weights satisfy every cover-dual constraint, checked exactly. Adding this dual to Strong+ narrows one interval and reduces the total width from 182 to 181, but changes no exact count or prescribed $M = 1, 4, 16$ decision. Against this stronger baseline, G8 removes eleven unresolved capacities in nine conditions and full refinement removes 65 in 43; the exact-count totals remain 103 and 114.

Protocol, checks and scope. This descriptive analysis reuses the small-instance set. Its protocol was specified before greedy execution; it is not an independent test population. Each input was run three times on one CPU thread. Every run regenerated the demand family, maximal completions, incidences, greedy library, price dual and exact checks. All 492 saved repetitions passed a separate auditor that reconstructed supports, replayed the exact remaining-coverage choices and ties, checked every request’s coverage, rebuilt the price dual, and checked all reported intervals. There were no greedy timeouts or failures. The median complete greedy pipeline took 1.138 ms per input, including enumeration and checks. Lower-certificate timings come from a separate execution, so no combined-portfolio runtime or speed ratio is inferred. Three repeats measure deterministic reproducibility and timing variation, not independent conditions.

M CONSERVATIVE DEMAND COUNTS AND THE LARGER-INSTANCE SET

The exploratory grid includes all 164 small-instance and 30 larger-instance conditions at $G = 4$, $Q = 64, 256, 1024$. The larger-instance set contains six first/middle/last block witnesses at $(E, k) = (24, 4), (48, 6)$ and 24 Digits conditions with $E = 64, k = 4, 6$. Digits (Alpaydin & Kaynak, 1998) uses scikit-learn’s 1,797-row test partition, all 64 integer pixels, parity targets, dictionary sizes 24 and 1,024, and $T = 16, 64$. Four dictionary streams supply 12 selected pairs, each at both cardinalities. These conditions share inputs and are not independent observations.

The count program retains exactly feasible full types, excludes impossible types, and computes one-sided rounded counts on partial types. It permits 200,000 type states and three million matrix entries, with a ten-second per-input cap. All 194 inputs and 582 resolution records finish, with no omitted condition. Integer arithmetic determines counts; numerical LP proposals undergo exact rational pricing against the original completion types.

The bounded-pool packing comparator draws 4,096 uniform k -subsets, retaining the first 512 distinct feasible masks and also recording the pool at 256 attempts. Insertion-greedy and all-start degree-ordered greedy searches supply checked cliques. Three SHA-derived seeds are evaluated for every input; the comparison below grants the control the best verified result over all seeds and both effort levels. This is bounded heuristic packing, not an optimal-packing claim or an equal-time comparison to conservative counts. The prefix-only G4 LP has no decision absent from packing at 4,096 attempts.

At $Q = 256$, twelve larger-input decisions exceed the packing control, but ten are explained by scalar or single-row information. The two remaining cases are the block witness with seed 1301, $E = 48, p = 5, k = 6$, certificate 70, and the Digits pair with dictionary size 1,024, $T = 64$, certificate 68, $k = 6$. Their joint bounds are respectively $244326247/163800 > 1024$ and $9413/126 > 64$. Same-count row bounds are $14471/20$ and $4925/126$; scalar bounds are $252689/6460$ and $22611385/6096454$; prefix-only bounds are $5566/105$ and $2408/627$; best checked packing sizes are 232 and 58. Packing supplies eight converse decisions across the larger-instance set.

An independent oracle enumerates group subsets and multiplies arbitrary-precision integer generating polynomials, rather than using the producer’s DP/convolution implementation. It checks all 6,318 count intervals, 309,363 type-pricing inequalities, and all 360 recorded packing cliques with

Inputs	Q	LP rejects	Packing/Strong+	Joint-only	Control-only
Small-instance 164	64	89	131	0	42
	256	93	131	0	38
	1024	93	131	0	38
Larger-instance 30	64	65	62	1	8
	256	66	62	2	8
	1024	66	62	2	8

Table 7: All condition–budget decisions at $M = 4, 16, 64, 256, 1024$. “Joint-only” additionally excludes decisions made by same-count scalar or strongest-row bounds and the prefix-only LP. “Control-only” counts Strong+ or packing decisions absent from the rounded LP. No matched-time comparison is implied.

Method at 1 s	$M = 4$	16	64	256	1024	Total/80
Fixed G4/G8; equal; 512-pool row/packing	16	13	9	0	0	38
Directed refinement + 512-pool packing	16	14	9	3	1	43
Same-path row + 512-pool packing	16	13	9	1	0	39
Uncapped streaming packing	15	13	12	9	0	49
Same-path row + uncapped packing	15	13	12	9	0	49

Table 8: Complete random-direction stress-test rejections. The last three controls were added after the prespecified comparison and are exploratory amendments. All methods have zero full-family upper brackets; counts retain all inputs.

466,393 conflicting pairs. Explicit enumeration checks the 164 small families; separate boundary fixtures exercise zero weights, exact thresholds and counts exceeding signed 64-bit range. These checks establish finite certificate validity, not generalization.

N RANDOM-DIRECTION STRESS TEST

Before generation, we fixed two sources, $E = 24, 48$ with $k = 4, 6$, two seeds per source, and two thresholds per menu: 16 conditions from eight menus. Gaussian menus use an $E \times 6$ standard-normal matrix, a standard-normal direction and independent coordinate scales uniform in $[.75, 1.25]$. Weights are squared scaled predictions, interpreted as exact dyadics. Digit-image menus use one sampled pair with different labels and E randomly selected coordinates from the 64 pixels; weights are exact squared differences. Seeds are 17017 and 17033 (Gaussian), 29009 and 29021 (Digits), hashed with source and dimension. The two thresholds are the floors of $(3/4)k\bar{w}$ and $(5/4)k\bar{w}$ after integer clearing. No failed or inapplicable input is replaced. These Gaussian directions are not optimized by the joint-risk oracle.

The six methods specified before generation use a one-second clock per condition and rotate order. Fixed G4/G8, equal refinement, ambiguity-directed refinement, LP-free row counting and packing are compared at 20, 100, 500 and 1,000 ms. Refinement starts at two sorted groups and ends at eight. For unstable demand it splits the group with largest summed demand multiplicity times its within-group prefix width over ambiguous types; for stable demand it weights completion widths by the robust primal. Ties favor larger groups then leftmost position. Groups split at their median. The bounded-pool fallback retains 512 sampled feasible batches. Every checkpoint includes only certificates whose exact checks finished before its time; no late result is backdated.

The 512-batch pool was inadequate as the sole large-budget control. An exploratory amendment adds uncapped insertion-greedy packing with exact incremental conflicts and a same-group LP-free path. All 16 row-only paths match their LP paths. Against their union, two LP-only outcomes remain: Gaussian $E = 48$, seed 17033, high threshold, $M = 1024$; and Digits $E = 24$, seed 29009, high threshold, $M = 16$. The controls supply ten converse outcomes. This is a posthoc mechanism check, not an independent confirmation of the selected two successes.

Exact posthoc diagnostics find only two inputs with $G_{DS} \leq 8$; the other 14 intrinsically exceed that cap. Checkpoint serialization failed on an absent upper, and an initial pure-streaming dispatch used

the bounded-pool sampler. Both failed executions remain in the record; the implementation was corrected and the unchanged inputs rerun. Independent audits check all 96 corrected prespecified-method records, 48 stronger-control records, 219 rational duals, 16,858 clique events and 576 checkpoints.

O OPTIMIZED-WITNESS EVALUATION PROTOCOL

The optimized-witness evaluation uses the sensing-task generator and joint-risk oracle. Before any task generation or method output, we fix seeds 41017, 41047, 41077 and 41113; shapes $(E, p, k) = (24, 3, 4), (48, 5, 6)$; Gaussian and exponential-decay sensing; diagonal error; $c = .65$ and target tolerance .002. This gives 16 tasks nested in eight seed-shape streams: the two families reset and share a random stream. The statistical interpretation corrects the protocol’s initial description of all 16 tasks as independent, without changing inputs or execution. Each task receives 20 seconds. Its distinct first, middle and last logged block certificates are selected without inspecting grouped bounds.

Generator and numerical environment. Gaussian tasks draw the unnormalized entries of A and a target vector a independently from $N(0, 1)$. Decay tasks sort E times drawn uniformly on $[.1, 2.5]$, use p geometrically spaced rates $r_j \in [.25, 3]$, and set $A_{ej} = \exp(-t_e r_j)$, $a_j = \exp(-3r_j)$. Each row of A is normalized in Euclidean norm, and a in ℓ_1 norm. The model domain is $[-1, 1]^p$ and $\Sigma = I$. NumPy’s generator uses `SeedSequence([seed, E, p, k])`, reset for each family; this is the shared-stream dependence above.

The grouping, counting and timed studies run on an Intel Xeon Platinum 8568Y+ Linux x86-64 host with 32 visible logical CPUs, Python 3.12.14, NumPy 2.3.5 and SciPy 1.17.0 with HiGHS 1.8.0. The evaluation uses one worker and one numerical thread; timings are host-specific. Upstream conic/MI optimization uses CVXPY 1.9.3, Clarabel 0.11.1, ECOS 2.0.14 and PySCIPOpt 6.2.1. Fixed selected inputs separate certificate replay from variation in time-limited generator trajectories.

For every selected difference, stored binary64 matrix and difference entries are treated as rationals, and $z = Ad$, $w_e = z_e^2$, $T = 4c^2$ are recomputed exactly. Checks require $|d_j| \leq 2$, feasible origin energy and target gap above its saved danger threshold. All 48 selected witnesses pass. Fourteen source tasks finish; two Gaussian $E = 48$ tasks cap but their selected witnesses remain. Numerical task optimization is not an interval proof; subsequent certificates are exact for the serialized rational inputs.

Eleven methods receive three seconds per witness and three rotated runtime repetitions: fixed G4/G8, equal refinement, directed refinement and its row-only control, canonical minimum/fill-eight LPs and the fill-eight row-only control, rounded $Q = 256$ G4 LP and its row-only control, and pure uncapped packing. Every arm uses the same uncapped insertion-greedy fallback and the same input-specific sampling stream. Row-only methods omit LP solves; rounded row-only includes all count construction. Directed row-only recomputes its own prefix-ambiguity scores without shared LP output. Once demand is stable, it replaces robust-primal weighting with largest-group median splits, breaking ties leftmost; its refinement path can therefore differ from the LP path. The method clock charges sorting, failed group queries, DP, LP proposals, sampling and exact verification. Imports, input/output and upstream task generation are excluded.

Checkpoints are 20, 100, 500, 1,000 and 3,000 ms. Only completed exact certificates count. Native LP calls have .15-second caps; prefix geometry permits 50,000 recursion nodes, 20,000 types and 500,000 matrix entries. Rounded geometry permits 200,000 states and three million entries. Capped group construction yields an abstention and then sampling within the remaining clock. No fractional upper is treated as an integer library. All conditions, caps, solver failures and all three runtimes remain; the best repetition is never selected.

An independent exact degree diagnostic, run outside the method clocks, finds $G_{DS} > 8$ on all 48 witnesses (range 10–48); eleven require 48 singleton groups. Thus this cohort cannot validate the small-instance set’s few-group applicability, even with an unlimited diagnostic clock.

Early refusal ablation. A separate posthoc protocol tests the cap-decision version of the same algorithm on all 164 small-instance, 16 random-direction stress-test and 48 optimized-witness eval-

Method	20 ms	100 ms	500 ms	1,000 ms	3,000 ms
Fixed G4	116–117	151–152	162	172	181
Fixed G8	70	141–143	162	171	181
Equal refinement	94–96	137–138	165–166	169–170	181
Directed LP	128–129	162–163	171	172	183
Directed row-only	110–111	146–147	162	166	181
Canonical minimum LP	63	70	115–116	127–136	180
Canonical fill-eight LP	63	70	115	132–134	180
Canonical fill-eight row-only	63	70	116	129–130	180
Rounded G4 LP	147	166	171	173–174	181
Rounded G4 row-only	137	159	165	171–172	181
Uncapped packing	120–121	149	160	171–172	181

Table 9: All 240 condition–budget queries at each checkpoint. Entries give the observed range across three runtime repetitions; a single number means all three agree. Every method produces zero full fractional upper brackets at every checkpoint. Canonical methods retain their fallback outcomes after a certified group-cap refusal.

uation inputs. It stops after eight witnessed necessary cuts, certifying that nine groups are required; no full partition or exact minimum is inferred from that return. Three rotated runs compare this with full minimum partitioning, with ten-second diagnostic caps. Every cut witness is checked directly; all accepted small-input partitions also pass an independent explicit-family invariance check. No run caps or changes the feasibility answer: the respective accepted/refused counts are 157/7, 2/14 and 0/48. On the 48 optimized-witness evaluation inputs, early refusal takes a median .148 ms, and its median within-input time ratio to full partitioning is .0372. The analogous random-direction stress-test ratio is .0530; on the small-instance set it is 1.003. These are diagnostic-only costs on reused inputs. This ablation does not replace the prespecified full-minimum arms of the three-second optimized-witness evaluation or imply an end-to-end sampler speedup.

All-method outcomes. All $48 \times 11 \times 3 = 1,584$ records complete. Table 9 reports all methods and checkpoints. At 100 ms the rounded LP contributes seven decisions beyond the union of same-count counting and uncapped packing, with none in the reverse direction, in all three repetitions. Those seven decisions span four of eight streams. At 20 ms the same comparison is nine versus seven; at 500 ms it is six versus zero; at one second it is two–three versus zero–one; at three seconds it is zero versus zero. Directed refinement gives 11–12 versus two at 100 ms, nine versus zero at 500 ms, and two versus zero at three seconds. These are paired condition–budget outcomes, not independent samples or discoveries of exact integer library sizes.

All rounded and fixed-G4 queries finish their geometric stages. Across repetitions, fixed G8 records 48 geometric caps, equal refinement 48, and each directed path 27; these are stage events, not excluded conditions. Each canonical path records 144 certified cap refusals. Independent checks recompute conservative counts, verify dual column loads and every stored packing witness, and reconstruct stable classes through exact compatible-coordinate degrees. A posthoc strengthening also combines a uniform per-demand price on all retained types with each single-row bound, using the same completed geometry and zero additional time charge. It changes some lower values but adds no budget decision at any checkpoint; the paired conclusions above are unchanged.

P CONTROLLED SIZE SWEEP

The controlled size sweep specifies 96 conditions before generation. For each $E \in \{12, 24, 48, 96\}$ and seed 51013, 51047, 51071 or 51109, NumPy’s `SeedSequence([seed, E])` generates $z \sim N(0, I_E)$. We square its binary64 entries as exact rationals, set $w_e = z_e^2$, and use $k \in \{2, 4, 6\}$ and $T \in \{3/4, 5/4\}$ $k \sum_e w_e / E$. Six conditions share each of 16 seed–size streams. These are controlled weight inputs, separate from the optimized-witness evaluation.

Thirteen methods receive one second: fixed G4/G8, directed LP/control, rounded G4 LP/control, pure packing, and early canonical LP/control pairs at caps 4, 8 and 16. Canonical acceptance uses largest-block median refinement up to $\min(G_{\max}, E)$; certified refusal switches to packing. Every

Method	20 ms	100 ms	500 ms	1 s	Full / within 1.25
Fixed G4	215	264	270	295–296	0 / 0
Fixed G8	125–127	251–253	271	296	0 / 0
Directed LP	223	273	289	297	9 / 7
Directed control	211	257–259	275	290–292	0 / 0
Rounded LP	220–223	287–288	292	300	0 / 0
Rounded control	219–220	283	287	298	0 / 0
Pure packing	223–224	264	268	296–297	0 / 0
Canonical cap 4 LP	224–225	265	269	298	4 / 0
Canonical cap 4 control	224	264	268	296–297	0 / 0
Canonical cap 8 LP	228–229	269	273	302	16 / 3
Canonical cap 8 control	224–225	265	269	297–298	0 / 0
Canonical cap 16 LP	192–193	269	273	302	30 / 26
Canonical cap 16 control	192–194	264	268	297	0 / 0

Table 10: All 480 condition–budget queries in the controlled size sweep. Each entry retains the range across three runtimes. The final column counts full fractional brackets and those with $U/D \leq 1.25$ at one second, out of 96 conditions. Blue identifies canonical full-bracket counts for comparison with fixed grouping; a bracket need not be tight. Controls omit LP solves and therefore supply no primal upper.

arm shares the input-seeded uncapped packing stream. Checkpoints are 20, 100, 500 and 1,000 ms, at the same five budgets as the optimized-witness evaluation. Geometry limits and .15-second native LP caps match that evaluation. Resource caps match within each LP/control pair; prefix and rounded geometry have different state representations and limits. Two workers use separate pinned CPUs with one numerical thread each; method order rotates over three runtime repetitions. A brief concurrent packaging check affected the CPU allocation. Before inspecting the affected outcomes, we selected a conservative six-condition time window and reran every method and repetition there; all pre-rerun records are retained.

Outside method clocks, minimum-partition diagnostics receive two seconds, with independent explicit degree checks for $E \leq 48$. All 24 $E = 12$ full fractional-cover references have matching exact rational lower and upper certificates. Method certificates are checked against these references. Independent checks also recompute grouped geometry and conservative counts, verify all duals and robust primals, and validate every stored packing clique and checkpoint.

All 3,744 method records pass these checks. The separate full diagnostic finishes on 88 conditions and caps on eight; none of the early cap tests times out. Caps 4/8/16 accept 4/16/41 conditions and refuse 92/80/55. At one second, the LPs yield 4/16/30 full brackets, of which 0/3/26 have $U/D \leq 1.25$. Eleven accepted cap-16 inputs hit a geometry limit. All 24 exact cap-16 brackets are singleton refinements at $E = 12$. The cap-16 full-bracket counts by $E = 12, 24, 48, 96$ are 24/4/2/0. Fixed G4/G8 give no full upper. Figure 4 separates intrinsic cap refusal from geometric resource failure and loose brackets.

Against the union of same-information counting and pure packing, rounded LP gives five LP-only versus zero control-only decisions at 500 ms, and two versus one at one second in every repetition. The latter LP-only counts by E are 1/0/1/0, spanning two streams; the converse is at $E = 48$. Directed LP gives fourteen versus zero at 500 ms, but four–five versus five at one second. Canonical cap 16 gives five versus zero at one second, all at $E \leq 24$. A posthoc uniform per-demand/per-row price strengthening, credited with zero extra time on each completed control geometry, adds no budget decision across 798 distinct geometries. These results delimit where joint certificates change decisions; they do not show broad scaling of tight full brackets.

Q SEARCH COMPARISON AND ITS LIMITS

The cover problem fixes one model difference. Search may select another one or use a direct optimization formulation, so a large cover is not a search lower bound. We report the corresponding negative controls.

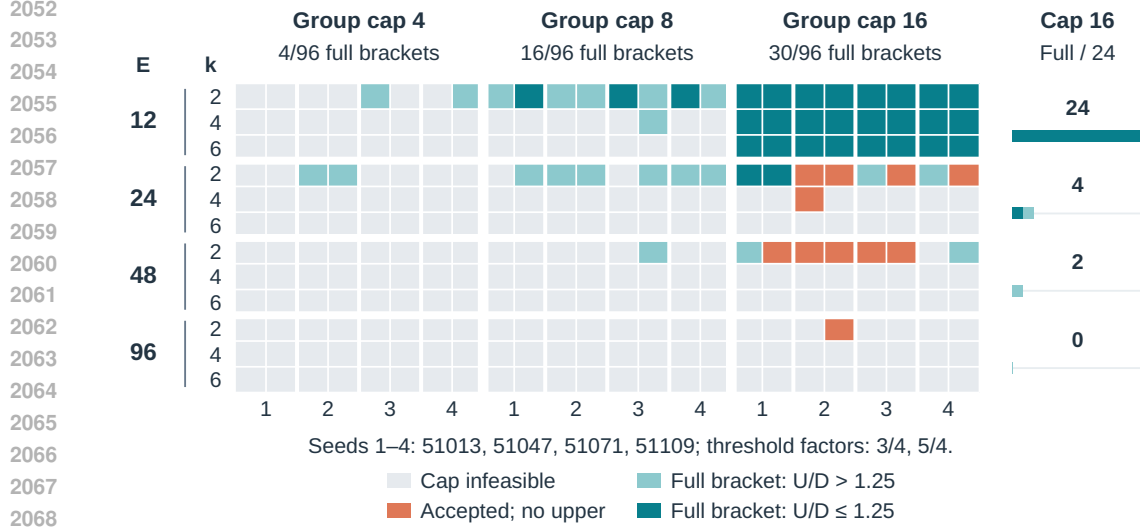


Figure 4: **Every condition and canonical group cap at one second.** Each cap retains all 96 conditions. Paired cells show threshold factors 3/4 then 5/4 within each seed. Gray certifies an insufficient group cap; coral means grouping succeeds but no full upper finishes. Light and dark teal mark full fractional brackets with $U/D > 1.25$ and $U/D \leq 1.25$. Headers count full brackets; right bars give cap-16 counts out of 24 for each E , split by the same bracket categories. All three repetitions agree. All conditions remain, including caps; a full fractional bracket is not an integer library.

The search comparison has 120 conditions: ten seeds, two $(p, k) = (3, 2), (5, 3)$ shapes at $E = 12$, two sensing families and three covariances (diagonal or adjacent blocks with correlations $\pm .8$). The families share seed–shape streams, giving 20 dependent clusters. Each method has a 30-second cap, cold caches, one numerical thread and tolerance .002. All 480 attempts in the common-wrapper comparison finish and agree with independent exhaustive optimum brackets.

Projection retains $(Ad, b^\top d)$ and broadcasts its lower bound to every compatible batch. Its completion control contracts the same direction to target gap $U - 3\varepsilon/4$ and uses block dynamic programming to maximize cancellation cardinality, keeping the origin bound separately. The direct sparse-dual control optimizes measurement support through the dual robust-risk formulation; its returned support is independently evaluated by the common conic oracle. Numerical MIP optimality is not taken as an exact certificate. Total method times are 17.26 s for projection, 38.94 s for this completion control and 7.36 s for direct optimization. Explicit oracle calls are 3,215, 7,393 and 120; the last count excludes internal MIP/conic work, which is charged to wall time. Thus direct optimization is the fastest.

A separate six-condition block-master study closes all three $E = 24$ conditions and none of the three $E = 48$ conditions under nominal ten-second caps. Repeated master calls consume 77.4–81.5% of elapsed time on the latter; direct SCIP closes four of six, including the larger diagonal case. An additional energy-minimizing direction selector on 12 reused conditions increases full-search calls from 116 to 150 and time from .84 to 1.65 s, despite stronger common-state coverage. This failed surrogate is not a method contribution. Complete generators, controls and execution settings are provided for reproduction, but their separate search theory is outside this paper’s fixed-family claims.

R THE COVERING OBJECT AND THE INHERITED TOOLS

For diagonal weights the compatible family is the cardinality- k layer of $\mathcal{K}(w, T) = \{C \subseteq [E] : \sum_{e \in C} w_e \leq T\}$. The completion problem covers this layer by principal downsets 2^C with $C \in \mathcal{K}(w, T)$. Equivalently, let $f(x) = \mathbf{1}\{w^\top x \leq T\}$ on Boolean vectors. Each legal completion yields the antimonotone term $t_C(x) = \bigwedge_{e \notin C} \neg x_e$. This is an implicant of f on the whole cube

because weights are nonnegative. Conversely a term of this form is globally sound only if its largest satisfying set C is feasible. Thus χ_k is the minimum number of these globally sound terms needed to cover the true points with $\sum_e x_e = k$.

Covering interpretations of formula size and lower bounds from points that share no implicant are established Boolean-representation tools (Hellerstein & Kletenik, 2013). Here two layer points share a legal term exactly when their union belongs to $\mathcal{K}(w, T)$, yielding the incompatibility-packing comparator. The fixed layer changes the covering task: we need not cover every true point on the cube, yet each selected term must remain a legal scenario globally. A general partial-function DNF may use terms that are not globally sound for this threshold function. Our lower bound does not apply to that less restricted representation.

Other reusable objects. Implicit hitting-set methods retain constraints and solve successive combinatorial subproblems (Moreno-Centeno & Karp, 2013; Ihalainen et al., 2026); logic-based Benders retains problem-specific cuts (Hooker & Ottosson, 2003). CEGIS(T) synthesizes symbolic programs with theory reasoning (Abate et al., 2023). Knowledge compilation and BDD representations can encode whole feasible families (Darwiche & Marquis, 2002; Abío et al., 2012). These objects need not be lists of concrete models. Our lower bounds therefore do not constrain their representation size or solve time.

Robust-design context. Worst-case input design and bounded-error identification motivate the underlying request geometry (Tanaskovic et al., 2014; Walz et al., 2018; 2020). Sensor-selection complexity and correlated-noise optimization address related design problems (Bajović et al., 2011; Liu et al., 2016). The present cover results specialize to a fixed pair and diagonal weights; the search comparison separately tests the cost of finding that pair. No new general covering relaxation or robust-design speedup follows from these connections.